PROCEEDINGS BOOK
of
GFSNP 2024

The 14th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications

EDITORS

Yılmaz ŞİMŞEK

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EDITION AND PUBLICATION DATE

First Edition

(March 28, 2024)

Dedicated to Professor Taekyun KIM

on the Occasion of his 60th Anniversary

Symposium Dates: January 18–21, 2024

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TITLE

Proceedings Book of 14th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP 2024)

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ON THE OCCASION OF HIS 60TH ANNIVERSARY

SYMPOSIUM VENUE

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Antalya, TURKEY

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ABOUT SYMPOSIUM

In 2009, the first edition of the Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP) was held in Rethymno, Crete Island, Greece, and it has been held regularly since 2009 as a minisymposia, in the partnership with the International Conference of Numerical Analysis and Applied Mathematics (ICNAAM). These all mini-symposiums were organized by Professor Dr. Yilmaz Simsek, and over the years, he gave the opportunity to the scientists working on generating functions for presenting their works. From the first day it was organized until today, this symposium series has brought together the researchers, who work on generating functions and related areas, from all over the world. Last year, in 2023, the 13th of the GFSNP symposium series was held in, Antalya, TURKEY, with dedication to Professor Dr. Yilmaz Simsek on the occasion of his 60th anniversary.

As for the 14th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP 2024), under the organization of Professor Yilmaz Simsek, it was held at Sherwood Exclusive Lara Hotel in Antalya, TURKEY, for four days from January 18 to January 21, 2024, by dedicating this wonderful symposium to the respected mathematician Professor Dr. Taekyun Kim on the occasion of his 60th anniversary.

The aim of the symposium GFSNP 2024 was to bring together leading scientists of the pure and applied mathematics and related areas to present their research, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other.

The symposium GFSNP 2024 took place in a hybrid form with both Physical and Virtual (Online) participations, for four days from January 18 to January 21, 2024. There were totally 66 participants from 10 different countries [Algeria, China, France, India, Iran, Morocco, Northern Cyprus, South Korea, Turkey, USA]. Among others, 3 of these participants attended the symposium as a listener, and 1 participant attended the symposium with a joint paper accompanied by one of the other participants. In addition, there were a total of 6 participants who made two presentations among others.

During the four days of GFSNP 2024, its participants totally made 66 presentations, and these contributions are respectively affiliated with 9 different countries [Algeria (4), China (5), France (1), India (3), Iran (1), Morocco (1), Northern Cyprus (6), South Korea (18), Turkey (27)].

In addition to a great number of excellent presentations, there was three listener participants who have supported our symposium by their presence.

As its name suggests, the theme of the symposium GFSNP 2024 is "*Generating Functions of Special Numbers and Polynomials and their Applications*". Considering that the generating functions has found a field of application in many different disciplines such as Algebra, Combinatorics, Number Theory, Analytic Number Theory, Graph Theory, Analysis of Algorithms, Mathematical Physics, Bio-informatics, Mathematical Chemistry, Mathematical Biology, Genetics, Management, Economics, Probability & Statistics, Engineering and their applications, we can easily state that the contents of oral and poster presentations of this symposium are mainly related to not only the generating functions, but also their applications in various fields of mathematics and related areas.

In this context, the contents of oral and poster presentations of this symposium were mainly related to not only the above areas, but also their applications in various fields of mathematics and related areas.

Further details about the symposium GFSNP 2024 are given as follows:

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- Mustafa Alkan, **Vice President**, (Akdeniz University, Turkey)
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- Irem Kucukoglu, (Alanya Alaaddin Keykubat University, Turkey)
- Ortaç Öneş, (Akdeniz University, Turkey)
- Ayse Yilmaz Ceylan, (Akdeniz University, Turkey)

INVITED SPEAKERS of GFSNP 2024

(Sorted list alphabetically by speaker's last name)

- **Abdelmejid BAYAD,** (Université d'Evry, FRANCE)
- **Ismail Naci CANGUL,** (Bursa Uludag University, TURKEY)
- **Taekyun KIM,** (Kwangwoon University, REPUBLIC of KOREA)
- **Veerabhadraiah LOKESHA,** (Vijayanagara Sri Krishnadevaraya University, INDIA)

Foreword written by Professor Taekyun Kim

It delivers me tremendous pleasure to write this foreword for the "*14th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications* (GFSNP 2024)" dedicated to my 60th birthday.

Perhaps the most special of these is the symposium dedicated to the age of researchers in mathematics or science.

I received my doctorate in 1994, studied number theory and special functions, and joined the Jangjeon Mathematics Society in 1996 to conduct research in this field.

In the study of number theory, the generating function is usefully used to derive useful interesting identity.

Recently, I performed various studies using generating functions of special functions. In particular, special properties related to probability were studied.

Since then, we know well that generating functions have been used in almost all fields of mathematics, Probability, Quantum Physics, Differential Equations Theory, other applied sciences, etc.

At this symposium, it was found that many researchers were using generating functions to study in various areas, and it was seen as a very interesting phenomenon.

I look forward to the endless and further development of these symposiums and I would like to thank my friend Professor Yilmaz Simsek and all organizing committee.

In addition, I would like to express my sincere gratitude to all participants and speakers.

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https://www.researchgate.net/profile/Taekyun-Kim-3

Foreword written by the Editors

With our most sincere respects to Professor Taekyun Kim, we are honored to write this foreword for the Proceedings of the *14th Symposium on the Generation and Applications of Functions of Special Numbers and Polynomials* (GFSNP 2024), dedicated to his 60th birthday.

Professor Taekyun Kim completed his doctoral thesis, titled "On the q-analogue of p-adic log-gamma and L-function", under the supervision of Professor Katsumi Shiratani at Kyushu University Japan between the 4th month of 1992 and the 3rd month of 1994. Since than he has continued to make invaluable contributions to number theory, p-adic analysis, and other areas of mathematics. On 6 January 1996, he founded the Jangjeon Mathematical Society: (`https://www.jangjeonopen.or.kr`). After that he became Founding Editor and Editors-in-Chief of the following journals:

- Advanced Studies in Contemporary Mathematics
- Proceeding of the Jangjeon mathematical Society.

As for his other scientific activities, he has more than 700 papers which are published in SCI/SCI-E/Scopus indexed Journals. He has received many awards. Furthermore, he has many Editorship and Associate Editorship positions in SCI/SCI-E/Scopus indexed Journals.

As for the symposium on "Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP)", let's give a brief history of the symposium. The 1st GFSNP symposium was first held in 2009 in Rethymno, Crete Island, Greece, by "Prof. Dr. Yılmaz Şimşek" in partnership with the organizer of ICNAAM, under the name of mini symposium. Later, despite the Coronavirus pandemic and other major earthquake disasters in our country, GFSNP continued to be held regularly every year by "Prof. Dr. Yılmaz Şimşek" and will continue to do so. Because today "Prof. Dr. Yılmaz Şimşek" has a very strong team. Thanks to this team, the GFSNP Symposium has taken its place among the world's respected symposiums and will continue to produce historical book series with ISBN numbers every year. Therefore, "Prof. Dr. Yılmaz Şimşek" would like to sincerely thank the following GFSNP's Editorial Board Team:

- "Prof. Dr. Mustafa Alkan", "Assoc. Prof. Dr. Irem Kucukoglu", "Assoc. Prof. Dr. Ortaç Öneş" and "Assoc. Prof. Dr. Neslihan Kilar".

And also "Prof. Dr. Yılmaz Şimşek" would like to express his sincere gratitude to the following organizing committee members:

- "Assoc. Prof. Dr. Ayşe Ceylan Yilmaz", "Dr. Buket Simsek", "Asst. Prof. Dr. Rahime Dere", "Assoc. Prof. Dr. Erkan Agyuz", "Assoc. Prof. Dr. Elif Cetin", "Dr. Damla Gün", "Dr. Busra Al", "Dr. Elif Sukruoglu", "Ezgi Polat", "Elif Bozo".

Thanks to these members of our team, over the years, GFSNP symposiums have expanded and given the opportunity to scientists working not only on generating functions and their applications, but also on the applications of these topics to all areas of mathematics, probability and statistics applications, engineering applications, and other branches of science, to present their work.

This symposium series, which covers the above topics since the first day it was organized, continues with a wide range of activities, expanding to bring together researchers working in relevant fields from all over the world.

As its name suggests, the topics of the symposium GFSNP are "Generating Functions of Special Numbers and Polynomials and Their Applications". Considering that generating functions find application in many different disciplines, such as Algebra, Combinatorics, Number Theory, Analytical Number Theory, Graph Theory, Analysis of Algorithms, Mathematical Physics, Bio-informatics, Mathematical Chemistry, Mathematical Biology, Genetics, Management, Economics, Probability & Statistics, Engineering, the contents of the oral and poster presentations of this symposium are mainly concerned with not only generating functions, but also their applications in various fields of mathematics and related fields.

Professor Taekyun Kim, who is the world's leading scientist, has been working together with Professor Yılmaz Şimşek since 2004. Therefore, this year the symposium GFSNP 2024 was organized, under the leadership of Professor Şimşek, with the theme of dedication to Professor Kim's 60th birthday.

As for the brief description about the contents of "Proceedings Book of the 14th the Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP 2024)" is given as follows:

The first section of the Proceedings Book of GFSNP 2024 includes the Ceremony talk of Professor Taekyun Kim, Foreword written by Professor Taekyun Kim, the Foreword written by the Editors, A brief biography of Professor Taekyun Kim, and some information about the symposium series GFSNP including the name of invited speakers, the name of committee members.

The rest of this book includes all the contributed talks and their related manuscripts.

In this regard, we would like to thank to all speakers and participants for their valuable contributions. We also express our sincere thanks to all members of the scientific committee and all members of the organizing committee because of their efforts to the success of this symposium and this book.

Once again, we would like to express that we are very happy that we celebrated the 60th birthday of very valuable and distinguished scientist Professor Taekyun Kim, and we also dedicate this book to his 60th birthday. For Professor Taekyun Kim's great contribution to mathematics, we would like to present our endless respect to him and to congratulate his 60th birthday with our best wishes one again. We wish the rest of his life to be happy, fruitful, success and he have many healthy years to spend with his loved ones.

We would also like to sincerely thank the following invited speakers who contributed to the realization of this symposium:

- Professor Abdelmejid Bayad (Université d'Evry, France),
- Professor Ismail Naci Cangul (Bursa Uludag University, Turkey),
- Professor Veerabhadraiah Lokesha (Vijayanagara Sri Krishnadevaraya University, India).

Finally, we would also like to express our sincere appreciation to everyone who contributed to the realization of this symposium.

Editors of the Proceedings Book of GFSNP 2024
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I. Brief Biography of Professor Taekyun Kim

Professor Taekyun Kim was born in Hapcheon, South Korea. He is married with Kyoung Young Lee and has two sons: Daehee and Changhee. There are many number and polynomial families in the literature with these names (Daehee and Changhee number and polynomials, etc.). As for his academic life, his PhD advisor is Prof. Dr. Katsumi Shiratani and Prof. Dr. Taekyun Kim received his PhD degree (in Number Theory in Mathematics) with Thesis Title: On the q-analogue of p-adic log-gamma and L-functions) at Kyushu University, Fukuoka of Japan between 1992–1994. Afterwards, Professor Kim's academic carrier was started in 1994 as a Researcher in Topology and Geometry Research Center at Kyungpook National University, and he has been working as Professor at Kwangwoon University since 2008. His research interests are "p-adic analysis, q-series, Special numbers and polynomials, Special Functions, Generating Functions, Special sums, Dedekind and Hardy Sums, Umbral Algebra, Umbral Analysis, etc.". According to the data in Google Scholar, he has publications more that 1000 and his works have received around **17768 citations** with **h-index 59** until now. In addition to writing hundreds of articles and book chapters in distinguished international journals of mathematical and engineering sciences, he has memberships of Editorial Board of many international journals. He has been invited to many scientific activities such as international conferences, seminars, visiting professor. He is referee and editor of many mathematical journals. His works contributed to many areas of mathematics. he has more than 700 papers which are published in SCI/SCI-E/Scopus indexed Journals. He has received many awards. Furthermore, he has many Editorship and Associate Editorship positions in SCI/SCI-E/Scopus indexed Journals.

In addition, on 6 January 1996, he founded the Jangjeon Mathematical Society: (`https://www.jangjeonopen.or.kr`). After that he became Founding Editor and Editor-in-Chief of the following journals:

- Advanced Studies in Contemporary Mathematics
- Proceeding of the Jangjeon mathematical Society.

II. Other Details

Education:

- 1991 – 1994: Kyushu University, Fukuoka, Japan (Doctor of Science, Number Theory in Mathematics).
- 1989 – 1991: Researcher student in Kyushu University, Fukuoka, Japan.
- 1987 – 1989: Kyungpook National University (Master of Science, Analysis in Mathematics).

- 1983 – 1987: Kyungpook National University (Bachelor).

Career:

- 2008/03/01 – Present: Professor, Department of Mathematics, Kwangwoon University, Seoul, Korea
- 2019/09/01 – 2026/08/30: Visiting Professor, Xi'an Technological University.
- 2015/09/01 – 2019/08/31: Chair Professor, Department of Mathematics, Tianjin Polytechnic University, Tianjin, China
- 2006/08/01 – 2008/02/29: Professor (BK), Department of Electrical Engineering and Computer Sciences, Kyungpook National University, Taegu, Korea
- 2001/04/12 – 2006/07/31: Research Professor, Department of Mathematics Education, Kongju National University, Kongju, Korea
- 1999/06/01 – 2000/12/31, Vistor, CECM, Simon Fraser University, Vancouver, Canada
- 1999/01/01 – 2000/02/29, Teacher(Math.), Korean Minjok Leadership Academy
- 1997/03/01 – 2007/02/28, Instructor, Republic of Korea Naval Academy
- 1994/04/01 – 1998/08/23, Instructor and Researcher, TGRC, Kyungpook National University, Taegu, Korea
- 2010/04/01 – 2012/03/31: The 10th Korea-Japan Basic Science Exchange Committee, Korea Research Foundation "The 10th Korea-Japan Basic Science Exchange Mathematics and Physics Division".

Editorship and Associate Editorship Duties:

- 2014/11/28 – 2021/12/31: Associate editor, Advances in Difference Equations
- 1999/01/01 – Present: Editor-in-chief, Advanced Studies in Contemporary Mathematics
- 2007/06/01 – Present: Editor-in-chief, Proceeding of the Jangjeon Mathematical Society
- 1996/06/01: Founder of Jangjeon Mathematical Society

Awards:

- 2013/05/20: Kwangwoon University Academy Award.
- 2014/12/04: Science Minister's Prize "Knowledge Creation Grand Prize".

- 2016/12: The world's most influential researcher (2016), Clarivate Analytics.
- 2017/11/27: 2017's Highly Cited Researcher, Clarivate Analytics (see, for details, `https://clarivate.com/hcr/2017-researchers-list/#freeText%3DKim%2C%20taekyun`).
- 2009/03: New hot paper-2009, Thomson Reuters (see, for details, `http://archive.sciencewatch.com/dr/nhp/2009/09maynhp/09maynhpKim`).

Further details about Professor Kim's professional and scholarly achievements, as well as honors and awards, can be found at the following websites:

- `https://www.kw.ac.kr/en/univ/science01_2.jsp`,
- `https://www.researchgate.net/profile/Taekyun-Kim-3`.

Ceremony Talk of Professor Taekyun Kim

Ladies and gentlemen,

Dear colleagues,

I would like to express my infinite gratitude to Professor Simsek and all committee members who prepared and held the international symposiumin connection with **my 60th birthday**, as well as to everyone involved in the symposium. I am also extremely happy to be participating in the international symposium held in Antalya, Turkey.

Additionally, I look forward to seeing you all again at the international academic conference held at Kwangwoon University in Korea this August. I'm not very expressive, so I can't say a long greeting, but it's a short sentence, but I hope that my infinite gratitude is expressed.

This year is the year of the blue dragon, the same year I was born. I end my greetings by wishing God's blessings on all attendees this year, **the Year of the Blue Dragon**.

Thank you.

Professor Taekyun Kim

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The Proceedings Book of the 14th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP 2024)

March 28, 2024

Contents

1 INVITED SPEAKERS

Some results on degenerate Fubini and degenerate Bell polynomials

Taekyun Kim

The aim of this paper is to further study some properties and identities on the degenerate Fubini and the degenerate Bell polynomials which are degenerate versions of the Fubini and the Bell polynomials, respectively. Especially, we find several expressions for the generating function of the sum of the values of the generalized falling factorials at positive consecutive integers.

2020 MSC: 05A15, 11B83

Keywords: Bell polynomials, Fubini polynomials, Generating function

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On the Jacobi forms and applications to theory of trigonometric sums

Abdelmejid Bayad

The Jacobi forms are a cross between elliptic functions and modular forms in one variable. Specifically, a Jacobi form on $SL_2(\mathbb{Z})$ is defined to be a holomorphic function

$$\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \quad (\mathbb{H} \text{ is the upper half-plane})$$

satisfying the two transformation equations

$$\phi\left(\frac{a\tau+b}{c\tau+b}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi(\tau, z)$$

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m\left(\lambda^2\tau+2\lambda z\right)} \phi(\tau, z),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$, and having a Fourier expansion of the form

$$\phi(\tau, z) - \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n, r) e^{2\pi i(n\tau+rz)}$$

k (resp. m) is called the weight (resp. index) of ϕ.

To a complex lattice L in $\mathbb{C}$, one can attach a Jacobi form denoted by $D_L(z, \varphi)$. This Jacobi form was studied extensively in [1]- [4], [6, 7]. Bayad and Gilles proved a distribution formula for $D_L(z, \varphi)$. Let Λ be a complex sublattice of $\mathbb{C}$ such that $L \subset \Lambda$. We have

$$D_\Lambda(z, \varphi) = \sum_{\bar{t} \in \Lambda / L} \mathbf{e}\left(-E_\Lambda(t, n\varphi)\right) D_L(z+t, n\varphi), \text{ (Distribution identity)}$$

where $n = [\Lambda : L]$. This identity is used to obtain results concerning quadratic Stickelberger elements (see [5, 6]).

In this talk, we prove a *generalized distribution identity* for the powers of $D_L(z, \varphi)$. One of the applications of this identity is to prove some new identities on the sums of powers of some trigonometric functions.

For instance, we show that

$$\frac{\pi n}{\sin(\pi n z)} = \sum_{k=0}^{n-1} (-1)^k \frac{\pi}{\sin\left(\pi z + \pi \frac{k}{n}\right)} \quad \text{for} \quad n = 3, 5, 7, \ldots, \tag{1}$$

and, we recover the well-known formula

$$\pi n \cot(\pi n z) = \sum_{k=0}^{n-1} \pi \cot\left(\pi\left(z+\frac{k}{n}\right)\right), \quad \text{for any } n \geq 1. \tag{2}$$

These two identities are straightforward applications of the distribution formula for the Jacobi form $D_L(z, \varphi)$.

We prove a generalized distribution identity for the powers for the powers of the Jacobi forms $D_L(z, \varphi)$, with applications to the theory of trigonometric sums. For more details see [3].

2020 MSC: 11F50

Keywords: Jacobi forms, Elliptic function

Acknowledgments

My talk is dedicated to Professor Taekyun KIM on the Occasion of his 60th Anniversary.

References

[1] A. Bayad, *Jacobi forms in two variables: Multiple elliptic Dedekind sums, the Kummer-Von Staudt Clausen congruences for elliptic Bernoulli functions and values of Hecke L-functions*, Montes Taurus J. Pure Appl. Math. **1 (2)**, 58–12, 2019.

[2] A. Bayad and A. Raouj, *Reciprocity laws for Dedekind cotangent sums*, Rocky Mountain J. Math. **53 (6)**, 1691–1707, 2023; DOI: 10.1216/rmj.2023.53.1691.

[3] A. Bayad and M. Hajli, *On Jacobi forms, and explicit evaluations of some trigonometric sums*, Ramanujan J. **60**, 27–57, 2023.

[4] A. Bayad and G. Robert, *Note sur une forme de Jacobi méromorphe*, C. R. Acad. Sci. Paris, Serie I **325**, 455–460, 1997.

[5] A. Bayad, W. Bley and Ph. Cassou-Noguès *Sommes arithmétiques et éléments de Stickelberger*, J. Algebra **179 (1)**, 145–190, 1996.

[6] A. Bayad and G. Robert, *Amélioration d'une congruence pour certains éléments de Stickelberger quadratiques*, Bull. Soc. Math. France **125N (2)**, 249–267, 1997.

[7] A. Bayad and G. Robert, *The Jacobi form D_L, theta functions, and Eisenstein series*, J. Théor. Nombres Bordeaux **28 (1)**, 75–88, 2016.

[8] M. Eichler and D. Zagier, *On the theory of Jacobi forms*, Progress in Math. (Volume 55), Birkhauser Verlag, Boston-Basel-Stuttgart, 1984.

[9] F. Hirzbruch, T. Berger and R. Jung, *Manifolds and modular forms*, Aspects of Mathematics, MPI, Bonn, 1992.

[10] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, (Ergeb. der Math. 88), Springer-Verlag, 1976.

[11] A. Weil, *Variétés kählériennes*, Paris, Hermann, 1958.

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On some new properties of omega invariant

Ismail Naci Cangul

We shall give some recent results on the number of realizations of a given degree sequence by means of omega invariant and also present some new property of omega invariant.

2020 MSC: 05C07, 05C10, 05C30

KEYWORDS: Degree sequence, Realizability, Omega invariant, Molecular graph

Introduction

In a tree, the number a_1 of the leaves (pendant vertices) is given by

$$a_1 = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta + 2.$$

Delen and Cangul rephrased the above formula as

$$a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = -2$$

and realized that for cyclic graphs, the left hand side of the above formula takes different values other than -2. These values are all even integers. To generalize the above formula so that it is also valid for the cyclic graphs, they defined the following invariant:

Definition 1. *Let $D = \left\{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\right\}$ be a set which also is the degree sequence of a graph G. The $\Omega(G)$ of the graph G is defined only in terms of the degree sequence as*

$$\begin{aligned} \Omega(G) = & \ a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 \\ = & \sum_{i=1}^{\Delta} (i-2)a_i. \end{aligned}$$

Since it was defined in 2018, over 60 papers are published on omega invariant and its properties. In [2], it was shown that for any graph G,

$$\Omega(G) = 2(m - n).$$

Hence for all graphs G, $\Omega(G)$ is even. Therefore for a randomly given set D of non-negative integers, if $\Omega(D)$ happens to be odd, then D is not realizable. Also the necessary and sufficient condition for a simple connected planar graph G to be a tree is shown to be $\Omega(G) = -2$.

Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be realizable as a graph G with c components. The number r of closed regions of G is given by $r = \frac{\Omega(G)}{2} + c$ in [3]. In the extremal problems, it is useful to know the minimum and maximum values of some quantity. When the number of components of all the realizations of a given degree sequence are in question, we have an exact lower bound for the number of components $c \geq -\frac{\Omega(G)}{2}$. This is clearly equivalent to the condition $c \geq n - m$. In [3], the maximum number of components of any realization of D might have is given by $c_{max} = \sum_{d_i \ even} a_i + \frac{1}{2} \sum_{d_i \ odd} a_i$.

Vertex and edge deletion are two useful operations on graphs. Using them successively, we can study some required property of a graph in terms of the same properties of some smaller graphs obtained by removing vertices or edges. We now give a formula which gives the amount of the change in omega invariant when a vertex is deleted from a graph:

Theorem 2. *Let G be a simple graph. Let v be a vertex of G of degree dv and let $G - v$ be the graph obtained by removing the vertex v from G. Then*

$$\Omega(G - v) = \Omega(G) - 2(d_G v - 1).$$

Proof. Recall that $\Omega(G) = 2(m - n)$ for a graph G of order n and size m. Let us denote by n' and m' the order and size of the graph $G - v$, respectively. Then $\Omega(G-v) = 2(m'-n')$. Once a vertex v is deleted, the number of vertices is reduced by 1 and the number of edges is reduced by $d_G v$. Therefore $m' = m - dv$ and $n' = n - 1$. Hence

$$\begin{aligned} \Omega(G - v) &= 2(m' - n') \\ &= 2(m - dv - (n - 1)) \\ &= 2(m - n) + 2(1 - d_G v) \\ &= \Omega(G) - 2(d_G v - 1). \end{aligned}$$

□

As omega invariant is closely related to the cyclomatic number of the graph which is the number of non-overlapping cycles, the following result is an immediate application of Theorem 2:

Corollary 3. *Let G be a simple graph. Let v be a pendant vertex of G of degree $d_G v = 1$ and let $G - v$ be the graph obtained by removing the vertex v from G. Then*

$$\Omega(G - v) = \Omega(G).$$

Proof. This is a special case of Theorem 2 where $d_G v = 1$. □

An alternative proof can be given as follows: When we delete a pendant vertex v from G, the cyclomatic number remains unchanged. So the result is clear.

Acknowledgments

This work is dedicated to the 60th birthday of our mutual friend Prof. Dr. Taekyun Kim.

References

[1] M. Ascioglu, M. Demirci and I. N. Cangul, *Omega invariant of union, join and corona product of two graphs*, Adv. Stud. Contemp. Math. **30 (3)**, 297–306, 2020.

[2] S. Delen and I. N. Cangul, *A new graph invariant*, Turkish J. Anal. Number Theory **6 (1)**, 30–33, 2018.

[3] S. Delen and I. N. Cangul, *Extremal problems on components and loops in graphs*, Acta Math. Sin. (Engl. Ser.) **35 (2)**, 161–171, 2019.

[4] S. Delen, M. Demirci, A. S. Cevik and I. N. Cangul, *On omega index and average degree of graphs*, J. Math. **2021**, 2021; Article ID: 5565146.

[5] S. Delen, M. Togan, A. Yurttas, U. Ana and I. N. Cangul, *The effect of edge and vertex deletion on omega invariant*, Appl. Anal. Discrete Math. **14 (3)**, 685–696, 2020.

[6] S. Delen, A. Yurttas, M. Togan and I. N. Cangul, *Omega invariant of graphs and cyclicness*, Appl. Sci. **21**, 91–95, 2019.

[7] M. Demirci, S. Delen, A. S. Cevik and I. N. Cangul, *Omega index of line and total graphs*, J. Math. **2021**, 2021; Article ID: 5552202.

[8] A. Y. Gunes, *Topological indices of dopamine*, Montes Taurus J. Pure Appl. Math. **2 (2)**, 63–68, 2020.

[9] M. S. Oz and I. N.Cangul, *Bounds for matching number of fundamental realizations according to graph invariant omega*, Proc. Jangjeon Math. Soc. **23 (1)**, 23–37, 2020.

[10] M. S. Oz and I. N. Cangul, *A survey of the maximal and the minimal nullity in terms of omega invariant of graphs*, Acta Univ. Sapientiae Math. **15 (2)**, 337–353, 2023.

[11] H. Ozden, F. Ersoy Zihni, F. Ozen Erdogan, I. N. Cangul, G. Srivastava and H. M. Srivastava, *Independence number of graphs and line graphs of trees by means of omega invariant*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**, 2020; Article ID: 91.

[12] U. Sanli, F. Celik, S. Delen and I. N. Cangul, *Connectedness criteria for graphs by means of omega invariant*, Filomat **34 (2)**, 647–652, 2020.

[13] M. Togan, A. Y. Gunes, S. Delen and I. N. Cangul, *Omega invariant of the line graphs of unicyclic graphs*, Montes Taurus J. Pure Appl. Math. **2 (2)**, 45–48, 2020.

[14] M. Togan, A. Yurttas Gunes, S. Delen and I. N. Cangul, *Omega invariant of the line graphs of unicyclic graphs*, Montes Taurus J. Pure Appl. Math. **2 (2)**, 45–48, 2020.

[15] A. Yurttas, M. Togan, S. Delen, U. Ana and I. N. Cangul, *The effect of a new graph theoretical invariant omega on some topological graph indices*, Creat. Math. Inform. **30 (2)**, 175–180, 2021.

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On the number of realizations of some degree sequences

Ismail Naci Cangul

The number of realizations of a given degree sequence is still an open problem. We shall give some new partial results on this realizability problem by means of recently defined and intensively studied omega invariant and its properties. We shall consider the simple, connected and unicyclic realizations of a given degree sequence with maximum cycle length 4 and also search for the simple, connected and unicyclic realizations with the unique cycle is a triangle. The total number of all these realizations in all possible cases are formulized.

2020 MSC: 05C07, 05C10, 05C30

KEYWORDS: Degree sequence, Realizability, Omega invariant, Unicyclic graph

Introduction

All graphs we shall consider will be simple and connected with n vertices and m edges. Delen and Cangul in [2] defined the following invariant:

Definition 1. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a set which also is the degree sequence of a graph G. The $\Omega(G)$ of the graph G is defined only in terms of the degree sequence as*

$$\Omega(G) = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = \sum_{i=1}^{\Delta}(i-2)a_i.$$

Omega invariant gives topological and combinatorial information about a graph or more importantly, about all realizations of a given degree sequence. These include many common properties like the number of faces, number of cycles which is the cyclomatic number, the possible lengths of the shortest and longest cycles which are very important in Eulerian and Hamiltonian problems. The motivation behind that idea was the formula for the number a_1 of the leaves (pendant vertices) in a tree which is given by $a_1 = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta + 2$. Delen and Cangul rephrased the above formula as

$$a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = -2$$

and observed that the LHS of the above equality takes different values other than -2 for cyclic graphs. Such values are interestingly even integers always.

Since it was defined in 2018, many papers are published on omega invariant and its properties, see e.g. [1, 4, 5], [7]- [13] for all these properties. In [2], it was shown that for any graph G, $\Omega(G) = 2(m - n)$. Hence for all graphs G, $\Omega(G)$ is even. Therefore for a randomly given set D of non-negative integers, if $\Omega(D)$ happens to be odd, then D is not realizable. Also the necessary and sufficient condition for a simple connected planar graph G to be a tree is shown to be $\Omega(G) = -2$.

Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be realizable as a graph G with c components. The number r of regions of G is given by $r = \frac{\Omega(G)}{2} + c$ in [3]. In the extremal problems, it is aimed to know the minimum and maximum values of some property. When the number of components of all the realizations of a given degree sequence are in question, we have an exact lower bound for the number of components $c \geq -\frac{\Omega(G)}{2}$. This is quivalent to the condition $c \geq n - m$. In [3], the maximum number of components of any realization of D might have is given by $c_{max} = \sum_{d_i \ even} a_i + \frac{1}{2} \sum_{d_i \ odd} a_i$.

It is well-known that a connected graph is unicyclic iff its omega invariant is zero. This is equivalent to the condition that the size m and the order n of the graph G are equal. The length of the unique cycle could be any integer between 1 and $n - a_1$ where a_1 is the number of pendant vertices in the graph. If the cycle has the maximum length, then we get the situation where all the pendant vertices are adjacent to the support vertices which are placed only on the unique cycle.

We give a partial result on the number of realizations of D:

Theorem 2. *Let $2 \leq i < j < k < l$. Let $D = \left\{1^{(a)}, i^{(1)}, j^{(1)}, k^{(1)}, l^{(1)}\right\}$. If $\Omega(D) = 0$, then all simple connected realizations are unicyclic and the maximum length of this unique cycle is 4. Also the total number of all simple connected unicyclic realizations (including those with a cycle of length 4 and those with a triangle) is 30 if $i = 2$ and 39 if $i > 2$.*

Proof. As the realization is asked to be simple connected and unicyclic, we have $\Omega(D) = 0$. Hence we obtain $a = i + j + k + l - 8$. First let $i = 2$. Then the other three support vertices will have degrees larger than 2. See Figure 1 for an illustration of the proof in the case that $i = 2, j = 3, k = 4$ and $l = 5$. First of all, the four integers $i = 2, j, k$ and l could be placed on the 4-gon as in $G1$ in $(4-1)!/2 = 3$ ways as this number is the total number of cyclic permutations. For each of these three placement, we have $3 + 2 + 2 + 2 = 9$ simple connected unicyclic realizations which have a triangle as the unique cycle. Note that all the graphs $G2 - G10$ are obtained from $G1$ by carrying one of the vertices on the 4-gon onto one of the pendant edges. Indeed the vertex of degree 2 can be carried onto a pendant edge in 3 ways and each of the other three non-pendant vertices can be carried onto a pendant edge in 2 ways. As for each of the three 4-gonal realizations, we get 9 triangular realizations, the total number of unicyclic realizations of D is $3 \times (1 + 9) = 30$.

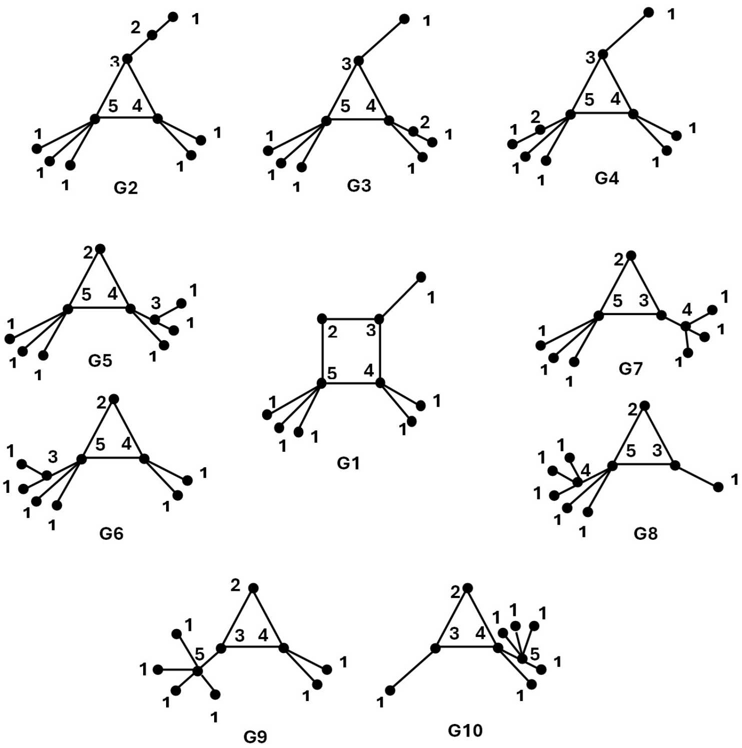

Figure 1: Graph $G1$ as a simple connected unicyclic realization of $D = \{1^{(6)}, 2^{(1)}, 3^{(1)}, 4^{(1)}, 5^{(1)}\}$ and all of its possible unicyclic realizations having a triangle

Let secondly $i > 2$. Then the other three support vertices will have degrees larger than 2. Similarly to the first case, we obtain the total number of simple connected and unicyclic realizations as $3 \times (1 + 12) = 39$. $\square$

Open problem: The calculations made for simple connected unicyclic realizations with maximum cycle length 4 in this paper can be extended to any maximum cycle length using similar methods. This could be an important step in finding a formula for the number of all graphs with a fixed and given number of vertices.

Acknowledgments

This work is dedicated to the 60th birthday of our mutual friend Prof. Dr. Taekyun Kim.

References

[1] M. Ascioglu, M. Demirci and I. N. Cangul, *Omega invariant of union, join and corona product of two graphs*, Adv. Stud. Contemp. Math. **30 (3)**, 297–306, 2020.

[2] S. Delen and I. N. Cangul, *A new graph invariant*, Turkish J. Anal. Number Theory **6 (1)**, 30–33, 2018.

[3] S. Delen and I. N. Cangul, *Extremal problems on components and loops in graphs*, Acta Math. Sin. (Engl. Ser.) **35 (2)**, 161–171, 2019.

[4] S. Delen, M. Demirci, A. S. Cevik and I. N. Cangul, *On omega index and average degree of graphs*, J. Math. **2021**; 2021, Article ID: 5565146.

[5] S. Delen, M. Togan, A. Yurttas, U. Ana and I. N. Cangul, *The effect of edge and vertex deletion on omega invariant*, Appl. Anal. Discrete Math. **14 (3)**, 685–696, 2020.

[6] S. Delen, A. Yurttas, M. Togan and I. N. Cangul, *Omega invariant of graphs and cyclicness*, Appl. Sci. **21**, 91–95, 2019.

[7] M. Demirci, S. Delen, A. S. Cevik and I. N. Cangul, *Omega index of line and total graphs*, J. Math. **2021**; 2021, Article ID: 5552202.

[8] A. Y. Gunes, *Topological indices of dopamine*, Montes Taurus J. Pure Appl. Math. **2 (2)**, 63–68, 2020.

[9] M. S. Oz and I. N. Cangul, *Bounds for matching number of fundamental realizations according to graph invariant omega*, Proc. Jangjeon Math. Soc. **23 (1)**, 23–37, 2020.

[10] M. S. Oz and I. N. Cangul, *A survey of the maximal and the minimal nullity in terms of omega invariant of graphs*, Acta Univ. Sapientiae Math. **15 (2)**, 337–353, 2023.

[11] H. Ozden, F. Ersoy Zihni, F. Ozen Erdogan, I. N. Cangul, G. Srivastava and H. M. Srivastava, *Independence number of graphs and line graphs of trees by means of omega invariant*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114**, 2020; Article ID: 91.

[12] M. Togan, A. Y. Gunes, S. Delen and I. N. Cangul, *Omega invariant of the line graphs of unicyclic graphs*, Montes Taurus J. Pure Appl. Math. **2 (2)**, 45–48, 2020.

[13] A. Yurttas, M. Togan, S. Delen, U. Ana and I. N. Cangul, *The effect of a new graph theoretical invariant omega on some topological graph indices*, Creat. Math. Inform. **30 (2)**, 175–180, 2021.

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New-fangled information of VL indices

Veerabhadraiah Lokesha

Connectivity indices are grown in the dissimilar areas of discrete structures as well as application in multidisciplinary aspects. More than ten thousand new Topological indices are developed and many put in the ground connected research activities are in bursting swing. The term Topological index is often reserved for graph invariant in discrete structures. In the Mathematical and Chemical literature, plenty of Topological Indices have been break new ground and extensively studied. The concept of VL index was recently pioneered by Deepika T in the Chemical graph theory. It is degree based Topological Indices in 2021. Motivated from this Suvarna and et al (2022) introduced the VL temperature index and status index for graph structures. These indices are well correlated to butane structure. In the 2023 winter, Deepasree S K and et al bring in the NVL index (neighbourhood VL indices). It is correction with heptanes isomers. Topological indices are used for quantitative structure-activity relationship (QSAR) and quantitative structure property relationship (QSPR) studies. Here I will be articulate and brushing the up to date enlargement of concatenation of VL indices.

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KEYWORDS: Topological indices, Status, Operators, Isomers

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On the fermionic and deformic p-adic q-integral formulas inspired by the papers of Taekyun Kim

Yilmaz Simsek

This presentation is inspired by the papers of Taekyun Kim, who not only constructed p-adic q-integrals involving the Volkenborn integral, but also give explicit formulas for a novel collection of generating functions for the special numbers and polynomials involving the Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Lah numbers, Peters numbers and polynomials, central factorial numbers, Daehee numbers and polynomials, Changhee numbers and polynomials, Harmonic numbers, Fubini numbers, Apostol-type numbers and polynomials, etc. The goal of this work is to survey formulas of p-adic q-integrals and their applications. Here we focus on the following works:

- T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 19, 288-299, 2002.
- T. Kim, On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$, J. Math. Anal. Appl. 331 (2), 779-792, 2007.
- Y. Simsek, Explicit formulas for p-adic integral: Approach to p-adic distributions and some families of special numbers and polynomials, Montes Taurus J. Pure Appl. Math. 1 (1), 1-76, 2019.

We give not only many formulas, but also further remarks and observations.

2020 MSC: 11S80, 11B68, 05A15, 05A19, 11M35, 30C15, 26C05, 12D10, 33C45

KEYWORDS: p-adic q-integrals, Volkenborn integral, Generating function, Special functions, Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Daehee numbers and polynomials, Changhee numbers and polynomials, Harmonic numbers, Combinatorial sums

Introduction, definitions and notations

Integral and derivative constitute the source of both science and today's modern life. Integral and derivative are at the center of creating mathematical models that directly enable the solution of real-world problems. Because area calculation, length calculation, volume calculation, optimization problems, creation of differential equations and finding their solutions etc. cannot be realized without integrals and derivatives. For this reason, for hundreds of years, mathematicians, physicists and other scientists have made very deep, effective and useful discoveries on derivative and integral calculus and their applications, and they still continue to be effective. Many different derivative concepts that are used effectively today have been defined. Similarly, many different integral concepts, definitions and methods have been found (and will continue to be found!). It will continue to be found because these definitions and concepts will not be the last stop of science. Integral and derivative concepts and structures will forever continue to be the pioneer and main source of scientific developments, and they will also shed light on unraveling the unknown mysteries of the universe.

The aim and motivation of this study is to summarize some types of integrals used today and to raise awareness about their effects. Perhaps this will give some researchers clues that will contribute to the development of their studies.

In particular, it is planned to present some applications of the p-adic integral(s) concepts and important formulas on how generating functions for some (special) numbers, polynomials and functions are found with these integrals techniques.

Integral

Integral is a concept generally associated with the derivative, which is the continuous analogue of a sum used to calculate arc lengths, areas, volumes, and their generalizations. The integral is also the most important mathematical model used to find the area under a curve, determine displacement from velocity, and solve other mathematical and physical problems. The concept of integral dates back to the 17th century by German mathematician Gottfried Wilhelm Leibniz (1 July 1646-14 November 1716) and English mathematician Sir Isaac Newton (25 December 1642-20 March 1727). These two mathematicians independently discovered the concept of integral using the fundamental theorem of mathematics. The development of the integral has been continues until today. For this reason, it is inevitable to find new types of integrals as well as new types of calculus based on needs (*cf.* [14, 79]). Some of them are briefly given by name as follows:

The **Darboux integral**, which is defined by Darboux sums (restricted Riemann sums) yet is equivalent to the Riemann integral. A function is Darboux-integrable if and only if it is Riemann-integrable. Darboux integrals have the advantage of being easier to define than Riemann integrals.

The **Riemann–Stieltjes integral**, an extension of the Riemann integral which integrates with respect to a function as opposed to a variable. The Riemann-Stieltjes integral is a generalization of the Riemann integral named after the German mathematicians Bernhard Riemann (826-1866) and Thomas Joannes Stieltjes (1856-1894). The Riemann-Stieltjes integral was first made by Thomas Joannes Stieltjes in 1894. There are very important relationships between this integral and the Lebesgue integral. This integral has very important applications both in mathematics and in physics, engineering and other branches of science, especially probability and statistics.

The **Lebesgue–Stieltjes integral**, further developed by Johann Radon, which generalizes both the Riemann-Stieltjes and Lebesgue integrals.

The **Daniell integral**, which subsumes the Lebesgue integral and Lebesgue-Stieltjes integral without depending on measures.

The **Cauchy's integral formula**, The Cauchy integral, named after the Frenchman Augustin-Louis Cauchy (21 August 1789-23 May 1857), is central to complex analysis. This integral allows finding results that are not valid in real analysis. It has very effective and vital applications.

The **Haar integral is defined by** Hungarian mathematician Alfréd Haar (11 October 1885-16 March 1933). This integral used for integration on locally compact topological groups, introduced by Alfréd Haar in 1933. This integral related to the Haar measure, which assigns an invariant volume to subsets of locally compact topological groups, consequently defining an integral for functions on those groups. The

Haar integral is also the general theory of Lebesgue integration. This integral define an integral for all Borel measurable functions.

The **Denjoy integral**, the **Luzin integral** or the **Perron integral**:

This equation, defined by the French mathematician Arnaud Denjoy (5 January 1884-21 January 1974), is today known as the Denjoy integral, Luzin integral or Perron integral. The Khinchin integral is a generalization of this integral, the Denjoy-Khinchin integral, the generalized Denjoy integral, or the extended Denjoy integral. The Denjoy integral is a generalization of the Riemann integral. Again, in some cases the Denjoy integral is more general than the Lebesgue integral.

The Henstock–Kurzweil integral is also known as generalized Riemann integral or the **Gauge integral**.

In particular, a function is Lebesgue integrable if and only if the function and its absolute value are Henstock-Kurzweil integrable.

The Henstock-Kurzweil integral, variously is defined by Arnaud Denjoy, Oskar Perron, and (most elegantly, as the gauge integral) Jaroslav Kurzweil, and developed by Ralph Henstock.

The **Itô integral and Stratonovich integral**, which define integration with respect to semimartingales such as Brownian motion. Itô (stochastic) integral on Itô calculus is shortly described as follows: In 1995, Itô calculus, named after Kiyosi Itô, who is Japanese mathematician (1915-2008), extends the methods of calculus to stochastic processes such as Brownian motion, related to Wiener process, on filtered probability space. This integration has important applications in mathematical finance and stochastic differential equations. The Itô stochastic integral is a stochastic generalization of the Riemann–Stieltjes integral.

The **Young integral**, which is a kind of Riemann-Stieltjes integral with respect to certain functions of unbounded variation.

The **rough path integral**, which is defined for functions equipped with some additional *rough path* structure and generalizes stochastic integration against both semimartingales and processes such as the fractional Brownian motion.

The **Choquet integral**, a subadditive or superadditive integral defined by French mathematician Gustave Choquet in 1953.

The **Bochner integral**, an extension of the Lebesgue integral to a more general class of functions. That is domain of the integrand of the Bochner integral is a Banach space (*cf.* [79]).

The **Motivic integration associated with arithmetic motivic measure**, in 1995, Maxim Kontsevich described Motivic integration, a concept in algebraic geometry. Later, this integral was developed to different dimensions by Jan Denef and François Loeser. Since its introduction, this integral has proven to be very useful in various branches of algebraic geometry, especially birational geometry and singularity theory. To put this roughly, motivational integration assigns to subsets of the arc space of an algebraic variety a volume that lives in the Grothendieck ring of algebraic varieties. The designation 'motive' reflects the fact that in motivational integration the values are geometric in nature, unlike in ordinary integration where the values are real numbers (*cf.* [74, 77]).

The p-adic integrals on *Ultrametric Calculus*, p-*adic analysis:*

Russian mathematician Kurt Hensel (1861-1941) defined p-adic numbers which are related to coding theory and Diophantine equations. By the aid of p-adic numbers, ultrametric calculus, p-adic analysis and other concepts can be constructed. p-adic numbers have been many applications in Number Theory, Algebraic Geometry, Algebraic Topology, Mathematical Analysis, Mathematical Physics, String Theory, Field

Theory, Stochastic Differential Equations on real Banach Spaces and Manifolds, p-adic distributions, p-adic measure, p-adic integrals, p-adic L-function, p-adic Analysis and Quantum Groups with Noncommutative Geometry, q-deformation of ordinary analysis etc. (*cf.* [4]- [68]). In order to survey and investigate the Volkenborn integral, the fermionic p-q-adic integral and also (p-adic) distributions, we give some notations and definitions.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively. Additionally, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $x \in \mathbb{R}$. The rising factorial and the falling factorial are defined as follows, respectively:

$$x^{(n)} = \begin{cases} x(x+1)(x+2)\dots(x+n-1) & \text{if} \quad n \in \mathbb{N} \\ 1 & \text{if} \quad n = 0 \end{cases} \tag{1}$$

and

$$x_{(n)} = \begin{cases} x(x-1)(x-2)\dots(x-n+1) & \text{if} \quad n \in \mathbb{N} \\ 1 & \text{if} \quad n = 0. \end{cases} \tag{2}$$

For $n \in \mathbb{N}_0$, we have

$$(-1)^n (-x)_{(n)} = (x+n-1)_{(n)} = x^{(n)} \tag{3}$$

(*cf.* [4]- [68]).

Generating functions for special numbers and polynomials

The Stirling numbers of the first kind $S_1(n,k)$ are defined by means of the following generating function:

$$F_{S1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{4}$$

and another generating function for the Stirling numbers of the first kind is falling factorial function which is given as follows:

$$x_{(n)} = \sum_{k=0}^{n} S_1(n,k)x^k \tag{5}$$

(*cf.* [4]- [68]).

The Stirling numbers of the second kind are defined by the following generating function:

$$\frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}; \qquad (k \in \mathbb{N}_0; \lambda \in \mathbb{C}) \tag{6}$$

and also

$$x^n = \sum_{k=0}^{n} S_2(n,k)x_{(k)}, \tag{7}$$

(*cf.* [4]- [68]).

Distributions and p-adic q-integrals on $\mathbb{Z}_p$

Let p be an odd prime number. Let $m \in \mathbb{N}$. Let $ord_p(m)$ denote the greatest integer k ($k \in \mathbb{N}_0$) such that p^k divides m in $\mathbb{Z}$. If $m = 0$, then

$$ord_p(m) = \infty.$$

Let $x \in \mathbb{Q}$, the set of rational numbers, with $x = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ with $n \neq 0$. Therefore,

$$\begin{aligned} ord_p(x) &= ord_p\left(\frac{a}{b}\right) \\ &= ord_p(a) - ord_p(b). \end{aligned}$$

Let $|.|_p$ is a map on $\mathbb{Q}$. This map, which is a norm over $\mathbb{Q}$, is defined by

$$|x|_p = \begin{cases} p^{-ord_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For instance, $x \in \mathbb{Q}$ with

$$x = p^y \frac{x_1}{x_2},$$

where $y, x_1, x_2 \in \mathbb{Z}$ and x_1 and x_2 are not divisible by p. Hence,

$$ord_p(x) = y$$

and

$$|x|_p = p^{-y}.$$

The set $\mathbb{Q}_p$ equipped with this norm $|x|_p$ is a topological completion of of set $\mathbb{Q}$. Let $\mathbb{C}_p$ be the field of p-adic completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{Z}_p$ be topological closure of $\mathbb{Z}$. Let $\mathbb{Z}_p$ be a set of p-adic integers, which is related to the norm $|x|_p$, given as follows:

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$

In order to define p-adic integral, the following definitions and formulas are needed: Let

$$f : \mathbb{Z}_p \to \mathbb{C}_p.$$

This function f is called a uniformly differential function at a point $a \in \mathbb{Z}_p$ if f satisfies the following conditions:

If the difference quotients

$$\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$$

such that

$$\Phi_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $f'(z)$ as $(x, y) \to (0, 0)$ (x and y remaining distinct). A set of uniformly differential functions is briefly indicated by $f \in UD(\mathbb{Z}_p)$ or $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$. The additive cosets of $\mathbb{Z}_p$ are given as follows:

$$p\mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : |x|_p < 1 \right\},\ 1 + p\mathbb{Z}_p, \ldots, p - 1 + p\mathbb{Z}_p,$$

where $p\mathbb{Z}_p$ is a maximal ideal of $\mathbb{Z}_p$ and for each $j \in \{0, 1, \ldots, p^n - 1\}$ we set

$$j + p^n\mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : |x - j|_p < p^{1-n} \right\}.$$

Thus, we have

$$\mathbb{Z}_p = \cup_{j=0}^{p-1} \left(j + p\mathbb{Z}_p \right).$$

Distributions:

Every map μ from the set of intervals contained in X to $\mathbb{Q}_p$ for which

$$\mu\left(x+p^n\mathbb{Z}_p\right)=\sum_{j=0}^{p-1}\mu\left(x+jp^n+p^{n+1}\mathbb{Z}_p\right)$$

whenever $x+p^n\mathbb{Z}_p\subset X$, exists uniquely to a p-adic distribution on X (*cf.* [2, 41, 54, 65, 70, 71]).

The Haar distribution is defined by

$$\mu_{Haar}\left(x+p^N\mathbb{Z}_p\right)=\mu_1\left(x+p^N\mathbb{Z}_p\right)=\frac{1}{p^N}, \tag{8}$$

which denotes by $\mu_1\left(x+p^N\mathbb{Z}_p\right)=\mu_1\left(x\right).$

The Dirac distribution is defined by

$$\mu_{Dirac}\left(x+p^N\mathbb{Z}_p\right)=\mu_\alpha\left(X\right)=\begin{cases}1 & \text{if } x\in X,\\ 0 & \text{otherwise.}\end{cases}$$

The Mazur distribution is defined by

$$\mu_{Mazur}\left(x+p^N\mathbb{Z}_p\right)=\frac{a}{p^N}-\frac{1}{2},$$

where $a\in\mathbb{Q}$ with $0\leq a\leq p-1$.

The Bernoulli distribution is defined by

$$\mu_{B,k}\left(x+p^N\mathbb{Z}_p\right)=p^{N(k-1)}B_k\left(\frac{a}{p^N}\right)$$

(*cf.* [2, 19, 40, 41, 54, 65, 70, 71]).

The distribution $\mu_{-1}\left(x+p^N\mathbb{Z}_p\right)$ on $\mathbb{Z}_p$ is defined by

$$\mu_{-1}\left(x+p^N\mathbb{Z}_p\right)=(-1)^x \tag{9}$$

(*cf.* [20, 22, 38, 39, 50]).

The Euler distribution is defined by

$$\mu_{\mathcal{E},k,q}\left(x+fp^N\mathbb{Z}_p\right)=(-1)^a\left(fp^{N)}\right)^k\mathcal{E}_k\left(\frac{a}{fp^N};q^{fp^N}\right),$$

where $N,k,f\in\mathbb{N}$ and f is odd (*cf.* [50, 46, 45, 63]). When $q\to1$, one has

$$\mu_{\mathcal{E},k,q}\left(x+fp^N\mathbb{Z}_p\right)\to(-1)^x\left(fp^{N)}\right)^kE_k\left(\frac{x}{fp^N}\right)$$

(*cf.* [50]).

p-adic q-integral

Kim [20] defined the p-adic q-integral as follows:

Let $f\in C^1(\mathbb{Z}_p\to\mathbb{K})$ and $q\in\mathbb{C}_p$ with $\left|1-q\right|_p<1$. Then we have

$$I_q(f(x))=\int_{\mathbb{Z}_p}f(x)d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}f(x)q^x, \tag{10}$$

where

$$[x] = [x:q] = \begin{cases} \frac{1-q^x}{1-q}, q \neq 1 \\ x, q = 1 \end{cases}$$

and

$$\mu_q(x) = \mu_q\left(x + p^N \mathbb{Z}_p\right)$$

which denotes q-distribution on $\mathbb{Z}_p$ and it is defined by

$$\mu_q\left(x + p^N \mathbb{Z}_p\right) = \frac{q^x}{[p^N]_q}$$

(*cf.* [20]).

When $q \to 1$, equation(10) reduces to the Volkenborn integral (bosonic integral)

$$\lim_{q\to 1} I_q(f(x)) = I_1(f(x)) = \int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{11}$$

where $\mu_1(x)$ is given by the equation (8), that is

$$\mu_1(x) = \frac{1}{p^N}$$

(*cf.* [2, 15, 54, 70, 71]); see also the references cited in each of these earlier works).

By using the Volkenborn integral (bosonic integral), generating functions for Bernoulli-type numbers and polynomials and other related special numbers and polynomials can be constructed (*cf.* [54], see also [20, 22, 38, 39, 40, 45, 46, 50, 63]).

When $q \to -1$, equation (10) reduces to the fermionic p-adic integral

$$\begin{aligned} \lim_{q\to -1} I_q(f(x)) = I_{-1}(f(x)) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \end{aligned} \tag{12}$$

where $\mu_{-1}(x)$ is given by the equation (9), that is

$$\mu_{-1}(x) = (-1)^x$$

(*cf.* [22], see also [38, 46, 45, 63]).

By using p-adic fermionic integral and its integral equations, generating functions for Bernoulli-type numbers and polynomials and other related special numbers and polynomials can be constructed (*cf.* [54], see also [20, 22, 38, 39, 40, 45, 46, 50, 63]).

Some properties of the Volkenborn integral (Bosonic p-adic integral)

Let

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \in C^1(\mathbb{Z}_p \to \mathbb{K}),$$

where

$$\binom{x}{n} = \frac{x_{(n)}}{n!}$$

the Mahler coefficients. Applying the Volkenborn integral to the above function $f(x)$ yields the following well-known formula:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} a_n,$$

(*cf.* [54, p. 168-Proposition 55.3]).

In [54], Schikhof gave the following integral formula for the Volkenborn integral:

$$\int_{\mathbb{Z}_p} f(x+n) d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{k=0}^{n-1} f'(k), \tag{13}$$

where

$$f'(x) = \frac{d}{dx}\{f(x)\}.$$

Let $f : \mathbb{Z}_p \to \mathbb{K}$ be an analytic function and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with $x \in \mathbb{Z}_p$.

The Volkenborn integral of this analytic function is given by

$$\int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n B_n$$

(*cf.* [20, 22, 54, 63]; see also the references cited in each of these earlier works).

Integral equation for the Volkenborn integral is given as follows:

$$\int_{\mathbb{Z}_p} E^m[f(x)]\, d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{j=0}^{m-1} \frac{d}{dx}\{f(x)\}|_{x=j}, \tag{14}$$

where

$$E^m[f(x)] = f(x+m)$$

and

$$\frac{d}{dx}\{f(x)\}|_{x=j} = f^{'}(j)$$

(*cf.* [20, 22, 54, 63, 76]; see also the references cited in each of these earlier works).

Using (10), the following integral equation was given by Kim [24]:

$$q \int_{\mathbb{Z}_p} E[f(x)]\, d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + \frac{q-1}{\log q} f^{'}(0) + (q-1) f(0) \tag{15}$$

(*cf.* see also [35]-[34]).

As usual, exponential function is defined as follows:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

The above series convergences in region $\mathbb{E}$ which is a subset of field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ (*cf.* [54, p. 70]).

Let k be residue class field of $\mathbb{K}$. If $char(k) = p$, then

$$\mathbb{E} = \left\{ x \in \mathbb{K} : |x| < p^{\frac{1}{1-p}} \right\},$$

and if $char(k) = 0$, then

$$\mathbb{E} = \{ x \in \mathbb{K} : |x| < 1 \}.$$

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. Kim [24, Theorem 1] gave the following integral equation:

$$\begin{aligned} & q^n \int_{\mathbb{Z}_p} E^n \left[f\left(x\right) \right] d\mu_q\left(x\right) - \int_{\mathbb{Z}_p} f\left(x\right) d\mu_q\left(x\right) \\ = \; & \frac{q-1}{\log q} \left(\sum_{j=0}^{n-1} q^j f^{'}(j) + \log q \sum_{j=0}^{n-1} q^j f(j) \right), \end{aligned} \tag{16}$$

where n is a positive integer.

Observe that substituting $n = 1$ into (16), we arrive at (15).

Theorem 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1\left(x\right) = \frac{(-1)^n}{n+1}. \tag{17}$$

Note that Theorem 1 was proved by Schikhof [54].

Substituting $m = 1$ and $f(x) = (1+a)^x$ into (14), we have

$$\int_{\mathbb{Z}_p} (1+t)^x d\mu_1\left(x\right) = \frac{1}{t} \log(1+t).$$

Therefore,

$$\sum_{n=0}^{\infty} t^n \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1\left(x\right) = \frac{1}{t} \log(1+t).$$

Combining the above equation with (17), we have the following well-known relation:

$$\log(1+t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

(*cf.* [54, 63, 76]). We observe that

$$\int_{\mathbb{Z}_p} a^x d\mu_1\left(x\right) = \frac{1}{a-1} \log_p(a),$$

where $a \in \mathbb{C}_p^+$ with $a \neq 1$ (*cf.* [54, p. 170]).

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. Then we have

$$\int_{\mathbb{Z}_p} f(-x) d\mu_1\left(x\right) = \int_{\mathbb{Z}_p} f(1+x) d\mu_1\left(x\right)$$

and if $f(-x) = -f(x)$, which is geometrically symmetric about the origin, we have

$$\int_{\mathbb{Z}_p} f(x) d\mu_1\left(x\right) = -\frac{1}{2} f'(0) \tag{18}$$

(*cf.* [54, p. 169]).

We also have the following well-known formula:

$$\int_{\mathbb{Z}_p} e^{ax} d\mu_1(x) = \sum_{n=0}^{\infty} B_n \frac{a^n}{n!}, \tag{19}$$

where $a \in \mathbb{E}$ with $a \neq 0$ (*cf.* [54, p. 172]). Using Taylor series for e^{ax} in the left-hand side of the equation (19), we have the following well-known the Witt's formula for the Bernoulli numbers of the first kind B_n:

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x), \tag{20}$$

where $n \in \mathbb{N}_0$ (*cf.* [54]; see also [20, 22] and the references cited in each of these earlier works).

By using (20) and $B_{2n+1} = 0$ with $n \geq 1$, we have

$$\int_{\mathbb{Z}_p} x d\mu_1(x) = B_1 = -\frac{1}{2},$$

$$\int_{\mathbb{Z}_p} x^{2n} d\mu_1(x) = B_{2n}$$

and

$$\int_{\mathbb{Z}_p} x^3 d\mu_1(x) = \int_{\mathbb{Z}_p} x^5 d\mu_1(x) = \cdots = \int_{\mathbb{Z}_p} x^{2n+1} d\mu_1(x) = 0.$$

Similarly, we have p-adic representation for the Bernoulli polynomials of the first kind as follows:

$$\int_{\mathbb{Z}_p} (z+x)^n d\mu_1(x) = B_n(z), \tag{21}$$

where $n \in \mathbb{N}_0$ and $B_n(z)$ denotes the Bernoulli polynomials, which are defined by means of the following generating function:

$$\frac{te^{tz}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!},$$

(*cf.* [3]- [67]; see also the references cited in each of these earlier works).

Let

$$P_n(x) = \sum_{j=0}^{n} \theta_j x^j$$

be a polynomial of degree n ($n \in \mathbb{N}_0$) and $\theta_j \in \mathbb{R}$. Substituting $P_n(x)$ into (11), we have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_1(x) = \sum_{j=0}^{n} \theta_j B_j$$

(*cf.* [63]). Putting $f(x,t;\lambda) = \lambda^x e^{tx}$ in (11), we have

$$\int_{\mathbb{Z}_p} \lambda^x e^{t(x+y)} d\mu_1(x) = \frac{(\log \lambda + t)\, e^{ty}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathfrak{B}_n(y;\lambda) \frac{t^n}{n!}, \tag{22}$$

where $\lambda \in \mathbb{Z}_p$ (*cf.* [37]; see also [11, 61, 68]). Using (22), we have

$$\int\limits_{\mathbb{Z}_p} \lambda^x (x+y)^n \, d\mu_1(x) = \mathfrak{B}_n(y;\lambda).$$

According to [17, 56, 72] and [39, 63], for each integer $N \geq 0$; C_{p^N} denotes the multiplicative group of the primitive p^Nth roots of unity in $\mathbb{C}_p^* = \mathbb{C}_p \setminus \{0\}$.

Let

$$\mathbb{T}_p = \left\{ \xi \in \mathbb{C}_p : \xi^{p^N} = 1, \text{ for } N \geq 0 \right\} = \cup_{N \geq 0} C_{p^N}.$$

In the sense of the p-adic Pontrjagin duality, the dual of $\mathbb{Z}_p$ is $\mathbb{T}_p = C_{p^\infty}$, the direct limit of cyclic groups C_{p^N} of order p^N with $N \geq 0$, with the discrete topology. $\mathbb{T}_p$ accept a natural $\mathbb{Z}_p$-module structure which can be written briefly as ξ^x for $\xi \in \mathbb{T}_p$ and $x \in \mathbb{Z}_p$. $\mathbb{T}_p$ are embedded discretely in $\mathbb{C}_p$ as the multiplicative p-torsion subgroup. If $\xi \in \mathbb{T}_p$, then $\vartheta_\xi : (\mathbb{Z}_p, +) \to (\mathbb{C}_p, .)$ is the locally constant character, $x \to \xi^x$, which is a locally analytic character if $\xi \in \{\xi \in \mathbb{C}_p : ord_p(\xi - 1) > 0\}$. Consequently, it is well-known that ϑ_ξ has a continuation to a continuous group homomorphism from $(\mathbb{Z}_p, +)$ to $(\mathbb{C}_p, .)$ (*cf.* [17, 39, 57, 56, 63, 72]; see also the references cited in each of these earlier works).

We assume that $\lambda \in \mathbb{T}_p$. Kim [37] defined the following integral:

$$\int\limits_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \mathcal{B}_n(\lambda), \tag{23}$$

where $\mathcal{B}_n(\lambda)$ denotes the Apostol-Bernoulli numbers, which are defined by means of the following generating function:

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda) \frac{t^n}{n!}, \tag{24}$$

(*cf.* [3, 67]).

Using (23) yields

$$\int\limits_{\mathbb{Z}_p} \lambda^x d\mu_1(x) = \frac{\log \lambda}{\lambda - 1}.$$

If $\lambda^k = 1$ with $k \in \mathbb{N}$, then we have

$$\int\limits_{\mathbb{Z}_p} \lambda^x d\mu_1(x) = 0.$$

This also gives us

$$\mathcal{B}_0(\lambda) = 0.$$

By using (23) and (24), we have

$$\int\limits_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \frac{n H_{n-1}(\lambda^{-1})}{\lambda - 1},$$

where $H_n(\lambda)$ denotes the Frobenious-Euler numbers, which are defined by means of the following generating function:

$$\frac{1-\lambda}{e^t - \lambda} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{t^n}{n!},$$

(*cf.* [55], see also [30], [37, Theorem 1, p. 439], [67, 68]).

The Volkenborn integral of some trigonometric functions are given as follows:

$$\int_{\mathbb{Z}_p} \cos(ax) d\mu_1(x) = \frac{a\sin(a)}{2(1-\cos(a))},$$

where $a \in \mathbb{E}$ with $a \neq 0$, $p \neq 2$ (*cf.* [54, p. 172], [23]);

$$\int_{\mathbb{Z}_p} \sin(ax) d\mu_1(x) = -\frac{a}{2},$$

where $a \in \mathbb{E}$ (*cf.* [54, p. 170], [23]); and also

$$\int_{\mathbb{Z}_p} \tan(ax) d\mu_1(x) = -\frac{a}{2}.$$

Note that

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_1(x) = \frac{1}{2}\int_{\mathbb{Z}_p} e^{ax} d\mu_1(x) - \frac{1}{2}\int_{\mathbb{Z}_p} e^{-ax} d\mu_1(x).$$

Combining the above equation with (19), we have

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_1(x) = -\frac{a}{2}.$$

p-adic integral over subsets of $\mathbb{Z}_p$

Let V be a compact open subset of $\mathbb{Z}_p$. Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. Then we have

$$\int_{V} f(x) d\mu_1(x) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x),$$

where

$$g(x) = \begin{cases} f(x) & \text{if } x \in V \\ 0 & \text{if } x \in \mathbb{Z}_p \setminus V \end{cases}$$

(*cf.* [54, p. 174]).

p-adic integral over cosets $j + p^n\mathbb{Z}_p$

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. Then we have

$$\int_{j+p^n\mathbb{Z}_p} f(x) d\mu_1(x) = \int_{p^n\mathbb{Z}_p} f(j+x) d\mu_1(x) = \frac{1}{p^n}\int_{\mathbb{Z}_p} f(j+p^n x) d\mu_1(x) \tag{25}$$

(*cf.* [54, p. 175]). Putting $f(x) = x^m$ ($m \in \mathbb{N}$) in (25) yields

$$\int_{j+p^n\mathbb{Z}_p} x^m d\mu_1(x) = p^{n(m-1)} B_m\left(\frac{j}{p^n}\right)$$

(*cf.* [54, p. 175]).

We now give some examples for the above formula:

Let

$$\mathbf{T}_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p$$

and $f : \mathbf{T}_p \to \mathbb{Q}_p$ and a C^1-function and also $f(-x) = -f(x)$ with $x \in \boldsymbol{T}_p$. Thus we have

$$\int\limits_{\boldsymbol{T}_p} f(x) d\mu_1(x) = 0.$$

Therefore

$$\int\limits_{\boldsymbol{T}_p} \frac{1}{x} d\mu_1(x) = \int\limits_{\boldsymbol{T}_p} \frac{1}{x^3} d\mu_1(x) = \int\limits_{\boldsymbol{T}_p} \frac{1}{x^5} d\mu_1(x) = \cdots = \int\limits_{\boldsymbol{T}_p} \frac{1}{x^{2n+1}} d\mu_1(x) = 0,$$

where $n \in \mathbb{N}$ (*cf.* [54, p. 175]) and

$$\int\limits_{\boldsymbol{T}_p} x^j \left(x^{p-1}\right)^s d\mu_1(x) = (j + (p-1)s)\, \zeta_{p,j}(s), \tag{26}$$

where $\zeta_{p,j}(s)$ denotes the p-adic zeta function, $|s|_p < p^{\frac{p-2}{p-1}}$, $s \neq -\frac{j}{p-1}$ and $j \in \{0, 1, \ldots, p-2\}$, $p \neq 2$ (*cf.* [54, p. 187], [68]).

Putting $s = n$ $(n \in \mathbb{N}_0)$ in (26), some values of the p-adic zeta function are given by

$$\int\limits_{\boldsymbol{T}_p} \left(x^{p-1}\right)^n d\mu_1(x) = \left(1 - p^{n(p-1)-1}\right) \frac{B_{n(p-1)}}{n(p-1)}$$

and

$$\int\limits_{\boldsymbol{T}_p} x^j \left(x^{p-1}\right)^n d\mu_1(x) = \left(1 - p^{j-1+n(p-1)}\right) \frac{B_{j+n(p-1)}}{j + n(p-1)}$$

whereas for $n \in \{2, 4, 6, 8 \ldots\}$, $j = 0$ and $p = 2$; and consequently we also have

$$\int\limits_{\boldsymbol{T}_p} x^n d\mu_1(x) = \left(1 - 2^{n-1}\right) \frac{B_n}{n}$$

(*cf.* [54, p. 187], [63, 68]).

p-adic integral of the falling factorial

Kim et al. [31] defined Witt-type identities for the Daehee numbers of the first kind by the following p-adic integral representation as follows:

$$D_n = \int\limits_{\mathbb{Z}_p} x_{(n)} d\mu_1(x), \tag{27}$$

where D_n denotes the Daehee numbers of the first kind, which are defined by means of the following generating functions, respectively:

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{28}$$

(*cf.* [52, p. 45], [31]). Using (28) yields

$$\begin{aligned} D_n &= \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) \\ &= (-1)^n \frac{n!}{n+1} \\ &= \sum_{v=0}^{n} S_1(n,v) B_v \end{aligned}$$

(*cf.* [31], see also [8, 51, 63]).

Theorem 2 (*cf.* [62]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2+3n+2} \tag{29}$$

$$= \sum_{k=1}^{n} S_1(n,k-1) B_k + B_{n+1}. \tag{30}$$

Theorem 3 (*cf.* [62]). *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_1(x) = (-1)^n H_n,$$

where H_n denotes the harmonic numbers given by

$$H_n = \sum_{k=0}^{n} \frac{1}{k+1}. \tag{31}$$

Theorem 4 (*cf.* [62]). *Let $n, r \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j}{n}^r. \tag{32}$$

Substituting $n = 1$ into (32), we get

$$\int_{\mathbb{Z}_p} x^r d\mu_1(x) = \sum_{k=0}^{r} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^r.$$

Combining the above formula with (20), we get the following theorem:

Theorem 5. *Let $r \in \mathbb{N}_0$. Then we have*

$$B_r = \sum_{k=0}^{r} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^r.$$

Theorem 6 (*cf.* [62]). *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x^m x_{(n)} d\mu_1(x) = \sum_{k=0}^{n} S_1(n,k) B_{k+m}. \tag{33}$$

Theorem 7 (*cf.* [62]). *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_1(x) = \sum_{j=0}^{n} \sum_{l=0}^{m} S_1(n,k) S_1(m,l) B_{j+l}, \tag{34}$$

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_1(x) = \sum_{k=0}^{m} (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{m+n-k+1}$$

and

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_1(x) = \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} k! \sum_{l=0}^{m+n-k} S_1(m+n-k,l) B_l.$$

Some properties of the fermionic p-adic integral

Here, we give some well-known properties of the fermionic p-adic integral.

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. Kim [23] gave the following integral equation for the fermionic p-adic integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p} E^n [f(x)] d\mu_{-1}(x) + (-1)^{n+1} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2 \sum_{j=0}^{n-1} (-1)^{n-1-j} f(j), \tag{35}$$

where $n \in \mathbb{N}$.

Substituting $n = 1$ into (35), we have very useful integral equation, which is used to construct generating functions associated with Euler-type numbers and polynomials, given as follows:

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0) \tag{36}$$

(*cf.* [23]).

By using (12) and (36), the well-known Witt's type formulas for the Euler numbers and polynomials of the first kind are given as follows, respectively:

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x), \tag{37}$$

where $n \in \mathbb{N}_0$ (*cf.* [9, 22]; see also the references cited in each of these earlier works) and $E_n(z)$ denotes the Euler polynomials, which are defined by following generating function

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

(*cf.* [3]- [67]), and

$$E_n(z) = \int_{\mathbb{Z}_p} (z+x)^n d\mu_{-1}(x), \tag{38}$$

where $n \in \mathbb{N}_0$ (*cf.* [9, 22]; see also the references cited in each of these earlier works) and $E_n(z)$ denotes the Euler polynomials, which are defined by following generating function

$$\frac{2e^{tz}}{e^t+1} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}$$

(*cf.* [3]- [67]).

Theorem 8 (*cf.* [35, Theorem 2.3])**.** *Let $n \in \mathbb{N}_0$. Then we have*

$$\int\limits_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x) = (-1)^n 2^{-n}. \tag{39}$$

By using the fermionic p-adic integral on $\mathbb{Z}_p$, Kim et al. [35] defined the Changhee numbers of the first as follows:

$$Ch_n = \int\limits_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x), \tag{40}$$

where Ch_n denotes the Changhee numbers of the first kind are defined by means of the following generating functions

$$\frac{2}{t+1} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \tag{41}$$

(*cf.* [35]). Using (41) yields

$$Ch_n = (-1)^n \frac{n!}{2^n} = \sum_{k=0}^{n} S_1(n,k) E_k \tag{42}$$

(*cf.* [35], see also [12, 47, 63]).

By using (35), Kim [24] modified (12) which gives the following integral equation:

$$q^d \int\limits_{\mathbb{Z}_p} E^d f(x)\, d\mu_{-q}(x) + \int\limits_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = [2] \sum_{j=0}^{d-1} (-1)^j q^j f(j), \tag{43}$$

where d is an positive odd integer.

The Volkenborn integral of some trigonometric functions are given as follows:

$$\int\limits_{\mathbb{Z}_p} \cos(ax) d\mu_{-1}(x) = 1,$$

where $a \in \mathbb{E}$ with $a \neq 0$, $p \neq 2$ (*cf.* [23]);

$$\int\limits_{\mathbb{Z}_p} \sin(a(x+1)) d\mu_{-1}(x) = -\int\limits_{\mathbb{Z}_p} \sin(ax) d\mu_{-1}(x),$$

where $a \in \mathbb{E}$ (*cf.* [23]); and also

$$(\cos(a)+1) \int\limits_{\mathbb{Z}_p} \sin(ax) d\mu_{-1}(x) = -\sin(a).$$

We also have

$$\int\limits_{\mathbb{Z}_p} \sinh(ax) d\mu_{-1}(x) = 1$$

(*cf.* [63]).

Let

$$P_n(x) = \sum_{j=0}^{n} a_j x^j$$

be a polynomial of degree n ($n \in \mathbb{N}_0$). Substituting $P_n(x)$ into (12), we have

$$\int\limits_{\mathbb{Z}_p} P_n(x) d\mu_{-1}(x) = \sum_{j=0}^{n} a_j \int\limits_{\mathbb{Z}_p} x^j d\mu_{-1}(x).$$

Since $E_{2n} = 0$ for $n \in \mathbb{N}$, by combining the above equation with (37), we thus have

$$\int\limits_{\mathbb{Z}_p} P_n(x) d\mu_{-1}(x) = 1 + \sum_{j=0}^{\left[\frac{n+1}{2}\right]} a_{2j+1} E_{2j+1}$$

(*cf.* [63]).

Substituting $g(x,t;\lambda) = \lambda^x e^{tx}$ into (36) yields

$$\int\limits_{\mathbb{Z}_p} g(x,t;\lambda) d\mu_{-1}(x) = \frac{2}{\lambda e^t + 1}.$$

Combining the above equation with (45) yields

$$\int\limits_{\mathbb{Z}_p} \lambda^x x^n d\mu_{-1}(x) = \mathcal{E}_n(\lambda), \tag{44}$$

where $\mathcal{E}_n(\lambda)$ denotes the Apostol-Euler polynomials $\mathcal{E}_n(\lambda)$ are defined by means of the following generating function:

$$\frac{2}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda) \frac{t^n}{n!}, \tag{45}$$

(*cf.* [6, 23, 63, 67]).

Integral formulas for the fermionic p-adic integral

Theorem 9 (*cf.* [62])**.** *Let $n \in \mathbb{N}$. Then we have*

$$\int\limits_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = (-1)^n \frac{(n-1)}{2^{n+1}} n!. \tag{46}$$

Theorem 10 (*cf.* [62])**.** *Let $n \in \mathbb{N}_0$. Then we have*

$$\int\limits_{\mathbb{Z}_p} x x^{(n)} d\mu_{-1}(x) = \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{(k-1)}{2^{k+1}} n!. \tag{47}$$

Theorem 11 (*cf.* [62])**.** *Let $n, r \in \mathbb{N}_0$. Then we have*

$$\int\limits_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j}{n}^r. \tag{48}$$

Substituting $n = 1$ into (48), we get

$$\int\limits_{\mathbb{Z}_p} x^r d\mu_{-1}(x) = \sum_{k=0}^{r} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^r.$$

Combining the above formula with (37), we get the following theorem:

Theorem 12. *Let $r \in \mathbb{N}_0$. Then we have*

$$E_r = \sum_{k=0}^{r} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^r .$$

Theorem 13 (*cf.* [62])**.** *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_{-1}(x) = \sum_{k=1}^{n} 2^{-k}. \tag{49}$$

With the aid of the following known formula

$$\sum_{k=1}^{n} x^k = \frac{x^{n+1}-x}{x-1},$$

equation (49) can be also given by the following theorem:

Theorem 14. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_{-1}(x) = 1 - 2^{-n}.$$

Theorem 15 (*cf.* [63])**.** *Let $m, n \in N_0$. Then we have*

$$\int_{\mathbb{Z}_p} x^m x_{(n)} d\mu_{-1}(x) = \sum_{k=0}^{n} S_1(n,k) E_{k+m}. \tag{50}$$

Theorem 16 (*cf.* [63])**.** *Let $m, n \in N_0$. Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_{-1}(x) = \sum_{k=0}^{m} (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{2^{m+n-k}}. \tag{51}$$

Combining (51) with (5), we get

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_{-1}(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} S_1(n,k) S_1(m,j) \int_{\mathbb{Z}_p} x^{k+j} d\mu_{-1}(x).$$

Combining the above formula with (37), we get the following theorem:

Theorem 17. *Let $m, n \in N_0$. Then we have*

$$\sum_{k=0}^{n} \sum_{j=0}^{m} S_1(n,k) S_1(m,j) E_{j+k} = \sum_{k=0}^{m} (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{2^{m+n-k}}.$$

Acknowledgments

This presentation and survey study is dedicated to my dear friend, brother and colleague Professor Taekyun KIM on the Occasion of his 60th Anniversary. I sincerely wish him to spend healthy and happy years with his family in his future life. At the same time, I sincerely hope that it will bring new concepts and theories to the field of mathematics.

References

[1] Y. Amice, *Mesures p-adiques, Séminaire Delange-Pisot-Poitou*, Théorie des nombres **6 (2)**, 1–6, 1964-1965.

[2] Y. Amice, *Integration p-adique Selon A. Volkenborn, Séminaire Delange-Pisot-Poitou*, Théorie des nombres **13 (2)**, G4, G1-G9, 1971-1972.

[3] T. M. Apostol, *On the Lerch zeta function*, Pacific J. Math. **1 (2)**, 161–167, 1951.

[4] S. Araci, M. Acikgoz abd E. Sen, *On the extended Kim's p-adic q-deformed fermionic integrals in the p-adic integer ring*, J. Number Theory **133 (10)**, 3348–3361, 2013.

[5] S. Araci, U. Duran and M. Acikgoz, *(p,q)-Volkenborn integration*, J. Number Theory **171**, 18–30, 2017.

[6] A. Bayad, Y. Simsek and H. M. Srivastava, *Some array type polynomials associated with special numbers and polynomials*, Appl. Math. Comput. **244**, 149–157, 2014.

[7] K. Iwasawa, *Lectures on p-adic L-function*, Princeton Univ. Press, Princeton, 1972.

[8] L.-C. Jang, W. Kim and H.-I. Kwon, *On degenerate Daehee polynomials and numbers of the third kind*, J. Comput. Appl. Math. 2019; https://doi.org/10.1016/j.cam.2019.112343.

[9] L. C. Jang and T. Kim, *A new approach to q-Euler numbers and polynomials*, J. Concr. Appl. Math. **6**, 159–168, 2008.

[10] L. C. Jang, T. Kim, D. H. Lee and D. W. Park, *An application of polylogarithms in the analogs of Genocchi numbers*, Notes Number Theory Discrete Math. **7 (3)**, 65–69, 2001.

[11] L. C. Jang and H. K. Pak, *Non-archimedean integration associated with q-Bernoulli numbers*, Proc. Jangjeon Math. Soc. **5 (2)**, 125–129, 2002.

[12] J. Jeong, D.-J. Kang and S.-H. Rim, *Symmetry identities of Changhee polynomials of type two*, Symmetry **10**, 2018; Article ID: 740.

[13] C. Jordan, *Calculus of finite differences* (2nd Edition), Chelsea Publishing Company, New York, 1950.

[14] B. V. Kallio, *A History of the definite integral (PDF) (M.A. thesis)*, University of British Columbia, Archived from the original on 2014-03-05, retrieved 2014-02-28, 1966.

[15] A. Khrennikov, *p-adic valued distributions and their applications to the mathematical physics*, Kluwer, Dordreht, 1994.

[16] N. Kilar and Y. Simsek, *A new family of Fubini type numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials*, J. Korean Math. Soc. **54 (5)**, 1605–1621, 2017.

[17] T. Kim, *An analogue of Bernoulli numbers and their congruences*, Rep. Fac. Sci. Engrg. Saga Univ. Math. **22 (2)**, 21–26, 1994.

[18] T. Kim, *The modified q-Euler numbers and polynomials*, 2007; ArXiv:math/0702523v1 (math.NT).

[19] T. Kim, *On a q-analogue of the p-adic log gamma functions*, J. Number Theory **76**, 320–329, 1999.

[20] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys. **19**, 288–299, 2002.

[21] T. Kim, *An invariant p-adic integral associated with Daehee numbers*, Integral Transforms Spec. Funct. **13 (1)**, 65–69, 2002.

[22] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integral and basic q-zeta function*, Trends Math. Information Center Math. Sciences **9**, 7–12, 2006.

[23] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on Z_p at $q = -1$*, J. Math. Anal. Appl. **331 (2)**, 779–792, 2007.

[24] T. Kim, *An invariant p-adic q-integral on $\mathbb{Z}_p$*, Appl. Math. Lett. **21**, 105–108, 2008.

[25] T. Kim, *p-adic l-functions and sums of powers*, 2006; ArXiv:math/0605703v1 (math.NT).

[26] T. Kim, *On the q-extension of Euler and Genocchi numbers*, J. Math. Anal. Appl. **326 (2)**, 1458–1465, 2007.

[27] T. Kim, *A note on central factorial numbers*, Proceed. Jangjeon Math. Soc. **21 (4)**, 575–588, 2018.

[28] D. Kim, H. O. Ayna, Y. Simsek and A. Yardimci, *New families of special numbers and polynomials arising from applications of p-adic q-integrals*, Adv. Difference Equ. **207**, 1–11, 2017.

[29] T. Kim, J. Choi, Y. H. Kim and C. S. Ryoo, *On the fermionic p-adic integral representation of Bernstein polynomials associated with Euler numbers and polynomials*, J. Inequal. Appl. **2010**, 20210; Article ID: 864247.

[30] D. S. Kim and T. Kim, *Some new identities of Frobenius-Euler numbers and polynomials*, J. Inequal. Appl. **307**, 1–10, 2012.

[31] D. S. Kim and T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. (Ruse) **7 (120)**, 5969–5976, 2013.

[32] D. S. Kim and T. Kim, *Some identities of Degenerate special polynomials*, Open Math. **13**, 380–389, 2015.

[33] T. Kim, D. S. Kim, D. V. Dolgy and J. J. Seo, *Bernoulli polynomials of the second kind and their identities arising from umbral calculus*, J. Nonlinear Sci. Appl. **9**, 860–869, 2016.

[34] T. Kim, M. S. Kim, L. C. Jang and S.-H. Rim, *New q-Euler numbers and polynomials associated with p-adic q-integrals*, 2007; ArXiv:0709.0089v1.

[35] D. S. Kim, T. Kim and J. Seo, *A note on Changhee numbers and polynomials*, Adv. Stud. Theor. Phys. **7**, 993–1003, 2013.

[36] D. S. Kim, T. Kim, J. J. Seo and T. Komatsu, *Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials*, Adv. Difference Equ. **92**, 1–16, 2014.

[37] T. Kim, S. H. Rim, Y. Simsek and D. Kim, *On the analogs of Bernoulli and Euler numbers, related identities and zeta and l-functions*, J. Korean Math. Soc. **45 (2)**, 435–453, 2008.

[38] M. S. Kim, *On Euler numbers, polynomials and related p-adic integrals*, J. Number Theory **129 (9)**, 2166–2179, 2009.

[39] M. S. Kim and J. W. Son, *Analytic properties of the q-Volkenborn integral on the ring of p-adic integers*, Bull. Korean Math. Soc. **44 (1)**, 1–12, 2007.

[40] M. S. Kim and J. W. Son, *Some remarks on a q-analogue of Bernoulli numbers*, J. Korean Math. Soc. **39 (2)**, 221–236, 2002.

[41] N. Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions* (Second Edition), Springer Verlag, New York, 1977.

[42] T. Kubota and H. W. Leopoldt, *Eine p-adische theorie der zetawerte I*, J. Reine Angew. Math. **214/215**, 328–339, 1964.

[43] S. Lang, *Cyclotomic fields*, Springer Verlag, New York, 1978.

[44] H. Ozden, *p-adic q-measure and its applications*, Doctoral Dissertation, Uludag University, Bursa, Turkey, 2009.

[45] H. Ozden and Y. Simsek, *Modification and unification of the Apostol-type numbers and polynomials and their applications*, Appl. Math. Comput. **235**, 338–351, 2009.

[46] H. Ozden, Y. Simsek and I. N. Cangul, *Euler polynomials associated with p-adic q-Euler measure*, Gen. Math. **15 (2-3)**, 24–37, 2007.

[47] J.-W. Park, G.-W. Jang and J. Kwon, *The λ-analogue degenerate Changhee polynomials and numbers*, Glob. J. Pure Appl. Math. **13 (3)**, 893–900, 2017.

[48] A. P. Prudnikov, Y. A. Bryckov and O. I. Maricev, *Integrals and series* (Volume 1), Elementary Functions, Nauka, Moscow, 1981, (in Russian), Translated from the Russian and with a Preface by N. M. Queen, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo and Melbourne, 1986.

[49] K. F. Riley, M. P. Hobson and S. J. Bence, *Mathematical methods for physics and engineering: A comprehensive guide* (Third Edition), Cambridge University Press, New York, 2006.

[50] S.-H. Rim and T. Kim, *A note on p-adic Euler measure on $\mathbb{Z}_p$*, Russ. J. Math. Phys. **13 (3)**, 358–361, 2006.

[51] S.-H. Rim, T. Kim and S. S. Pyo, *Identities between harmonic, hyperharmonic and Daehee numbers*, J. Inequal. Appl. **2018**, 2018; Article ID: 168.

[52] J. Riordan, *Introduction to combinatorial analysis*, Princeton University Press, 1958.

[53] A. M. Robert, *A course in p-adic analysis*, Springer, New York, 2000.

[54] W. H. Schikhof, *Ultrametric calculus: An introduction to p-adic analysis*, Cambridge Studies in Advanced Mathematics 4, Cambridge University Press, Cambridge, 1984.

[55] I. J. Schoenberg, *Cardinal interpolation and spline functions IV. The exponential Euler splines*, In: Linear Operators and Approximation/Lineare Operatoren und Approximation (Ed. by P.L. Butzer, J. P. Kahane, B. Szökefalvi-Nagy), International Series of Numerical Mathematics/Internationale Schriftenreihe zur Numerischen Mathematik/Série Internationale D'Analyse Numérique (Volume 20), Birkhäuser, Basel, Switzerland, 1972.

[56] K. Shiratani and S. Yokoyama, *An application of p-adic convolutions*, Mem. Fac. Sci. Kyushu Univ. Ser. A Math. **36 (1)**, 73–83, 1982.

[57] Y. Simsek, *Twisted p-adic* (h, q)*-L-functions*, Comput. Math. Appl. **59 (6)**, 2097–2110, 2010.

[58] Y. Simsek, *Special numbers on analytic functions*, Appl. Math. **5 (7)**, 1091–1098, 2014.

[59] Y. Simsek, *Apostol type Daehee numbers and polynomials*, Adv. Studies Contemp. Math. **26 (3)**, 1–12, 2016.

[60] Y. Simsek, *Analysis of the p-adic q-Volkenborn integrals: An approach to generalized Apostol-type special numbers and polynomials and their applications*, Cogent Math. **3**, 1–17, 2016.

[61] Y. Simsek, *Identities on the Changhee numbers and Apostol-Daehee polynomials*, Adv. Stud. Contemp. Math. **27 (2)**, 199–212, 2017.

[62] Y. Simsek, *Formulas for p-adic q-integrals including falling-rising factorials, combinatorial sums and special numbers*, 2017; ArXiv:1702.06999v1 (math.NT).

[63] Y. Simsek, *Explicit formulas for p-adic integral: Approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 1–76, 2019.

[64] Y. Simsek, *Peters type polynomials and numbers and their generating functions: Approach with p-adic integral method*, Math. Methods Appl. Sci. **42 (18)**, 7030–7046, 2019; Doi: 10.1002/mma.5807.

[65] Y. Simsek and H. M. Srivastava, *A family of p-adic twisted interpolation functions associated with the modified Bernoulli numbers*, Appl. Math. Comput. **216 (10)**, 2976–2987, 2010.

[66] Y. Simsek, S. H. Rim, L.-C. Jang, D. J. Kang and J. J. Seo, *A note on q-Daehee sums*, Proc. 16th Int. Conf. Jangjeon Math. Soc. **16**, 159–166, 2005.

[67] H. M. Srivastava and J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[68] H. M. Srivastava, T. Kim and Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys. **12**, 241–268, 2005.

[69] B. A. Tangedal and P. T. Young, *On p-adic multiple zeta and log gamma functions*, J. Number Theory **131 (7)**, 1240–1257, 2011.

[70] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-adic analysis and mathematical physics*, World Scientific, Singapore, 1994.

[71] A. Volkenborn, *On generalized p-adic integration*, Mém. Soc. Math. Fr. **39-40**, 375–384, 1974.

[72] C.F. Woodcock, *Convolutions on the ring of p-adic integers*, J. Lond. Math. Soc. **20 (2)**, 101–108, 1979.

[73] G. E. Wahlin, *A new development of the theory of algebraic numbers*, Trans. Amer. Math. Soc. **16 (4)**, 502–508, 1915.

[74] W. Veys, *Arc spaces, motivic integration and stringy invariants*, Available at https://perswww.kuleuven.be/~u0005725/articles/2006/001 /ArcSpacesMotivicIntegrationAndStringyInvariants.pdf

[75] en.wikipedia.org/wiki/Falling_rising_factorials, (Accession date: 6 Jan 2024).

[76] en.wikipedia.org/wiki/Volkenborn_integral, (Accession date: 6 Jan 2024).

[77] https://en.wikipedia.org/wiki/Motivic_integration, (Accession date: 6 Jan 2024).

[78] https://en.wikipedia.org/wiki/It%C3%B4_calculus, (Accession date: 6 Jan 2024).

[79] https://en.wikipedia.org/wiki/Integral, (Accession date: 6 Jan 2024).

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2 CONTRIBUTIONS

Remarks on integral formula generated by Hurwitz-Lerch zeta function with order 1

Aykut Ahmet Aygunes

We [4] gave an integral formula associated with Hurwitz-Lerch zeta function with order 1 which is special case of Hurwitz-Lerch zeta function. Then by using this formula, he obtained a corollary generated by integral representation of Hurwitz-Lerch zeta function with order 1. In this presentation, we investigate and survey on unified formula by means of the result of our formulas and zeta function [4].

2020 MSC: 11M35, 30B10

KEYWORDS: Hurwitz-Lerch zeta function, Riemann zeta function, Hurwitz-Lerch zeta function with order 1

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References

[1] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, Heidelberg and Berlin, 1976.

[2] A. A. Aygunes, *Some identities for a special case of Hurwitz-Lerch zeta function*, AIP Conf. Proc. **1863 (1)**, 2017.

[3] A. A. Aygunes, *Hurwitz-Lerch zeta function with order 1*, J. Math. Sci. **1**, 16–24, 2017.

[4] A. A. Aygunes, *An integral formula generated by Hurwitz-Lerch zeta function with order* 1, Montes Taurus J. Pure Appl. Math. **3 (3)**, 12–16, 2021.

[5] A. A. Aygunes, *A new type integral formula derived from Hurwitz-Lerch zeta type function*, Montes Taurus J. Pure Appl. Math. **5 (2)**, 66–70, 2023.

[6] A. Erdélyi, *Higher transcendental functions* (Volume 1), McGrow-Hill, New York, 1955.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental tunctions* (Volume 1), McGraw-Hill Book Company, New York, Toronto and London, 1953.

[8] J. Choi, *Remark on the Hurwitz-Lerch zeta function*, Fixed Point Theory Appl. **70**, 2–10, 2013.

[9] L. Lewin, *Polylogarithms and associated functions*, New York, North-Holland, 1981.

[10] E. C. Titchmarsh, *The theory of the Riemann-zeta function*, Oxford University (Clarendon) Press, Oxford and London, 1951; Second Edition (Revised by D. R. Heath-Brown), 1986.

[11] H. M. Srivastava, *Some properties and results involving the zeta and associated functions*, Funct. Anal. Approx. Comput. **7 (2)**, 89–133, 2015.

[12] H. M. Srivastava and J. Choi, *Series associated with the zeta and related functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

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Investigating analytical bounds for the vertex corona in simple connected graphs

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Topological indices play a crucial role in the analysis of chemical graphs, offering numerical insights into compound and network structures. Widely utilized in chemical modeling to quantify relationships in structure-activity-property-toxicity and combinatorial library screening, this article presents significant discoveries. By establishing lower and upper bounds for various indices based solely on the minimum and maximum degree values within the graph denoted as G, the study provides a simplified approach for assessing topological indices in chemical compounds.

2020 MSC: 05C07, 05C90, 05C92, 05C12

Keywords: Corona Graph, Vertex corona graph, Lower bounds, Upper bounds and Topological index

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Observations on generalized Pöschl-Teller potential related to dimer interaction

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In this presentation we study on generalized Pöschl-Teller potential associated with dimer interaction. All parameters of the generalized Pöschl-Teller potential are determined by fitting Morse potential. With the aid of parameters, the energy levels of different diatomic molecules ranging are determined. It is observed that the potential provides a good agreement to experimental data as well as previous theoretical works conducted for the same dimer systems in the literature.

Furthermore, we give many new results and applications on the generalized Pöschl-Teller potential with the dimer interaction.

2020 MSC: 35J10

KEYWORDS: Pöschl-Teller potential, Diatomic interactions, Morse potential, Mathematical physics

Introduction

It is important to determine inter-atomic potentials to understand behaviour of molecules. Some potentials have been widely used for this purpose such as Lennard-Jones potential, Morse potential, Buckingham potential(electrostatic), Born-Karman potential etc. The Pöschl-Teller potential is a mathematical model used to describe behaviour certain molecules in quantum mechanics. The potential, depends on the inter-molecular distance, allows for the analysis of molecular properties such as bond dissociation energies and vibrational frequencies. Because of having analytically solvable function and including adjustable parameters make it convenient. In this study, we can use the following methods which were introduced in [1] and [2].

Schrödinger equation is given by (*cf.* [1]):

$$\frac{d^2\Psi(r)}{dr^2} + (E - V(r))\,\Psi(r) = 0. \tag{1}$$

If the equation is solved and rearranged (*cf.* [2]):

$$V(r) = -\frac{A\exp(-2ar)}{(1+b^2exp(-2ar))^2} + \frac{B\exp(-2ar)}{(1-b^2exp(-2ar))^2} \tag{2}$$

which named Generalized Pöschl-Teller Potential, and energy spectra is obtained:

$$E_n = -a^2\left(2n+1-\frac{1}{2}\sqrt{1+\frac{A}{a^2b^2}}+\frac{1}{2}\sqrt{1+\frac{B}{a^2b^2}}\right)^2. \tag{3}$$

For two atomic molecules, the potential can be determined by three conditions:

$$\begin{aligned} V'(r_e) &= 0, \\ V(\infty) - V(r_e) &= D_e, \\ V''(r_e) &= k_e \end{aligned} \tag{4}$$

and from the rotational-vibrational coupling constants;

$$\alpha_e = -\frac{6B_e^2}{\omega_e}\left(\frac{X_{r_e}+3}{3}\right) = F\left(\frac{6B_e^2}{\omega_e}\right), \tag{5}$$

where w_e is the vibrational frequency, B_e rotational constant.

$$\Delta = \frac{k_e r_e^2}{2D_e},$$

Sutherland parameter, k_e is force constant, r_e is equilibrium distance between two atoms, D_e is dissociation energy, the quantity

$$\Gamma = \frac{1}{9}X^2 r_e^2,$$

where

$$X = \frac{V'''(r_e)}{V''(r_e)},$$

where

$$\frac{d^2V}{dr^2} = V''$$

and

$$\frac{d^3V}{dr^3} = V'''$$

and unharmonicity is given by

$$w_e X_e = 8\Delta\frac{W}{r_e^2\mu_A}.$$

Thus, we have

$$\begin{aligned} y_e^2 &= \frac{\pm\sqrt{\frac{\Gamma}{\Delta}}-1}{1\pm\sqrt{\frac{\Gamma}{\Delta}}}, \\ a &= \pm\frac{\sqrt{\Delta}}{2r_e}, \\ b^2 &= y_e e^{\pm\Delta}; \end{aligned}$$

and also

$$\begin{aligned} B &= \frac{D_e b^2(1-y_e)^4}{4y_e^2}, \\ A &= \left(\frac{1+y_e}{1-y_e}\right)^4 B \end{aligned}$$

(*cf.* [2]).

Conclusion

In this study, we studied and surveyed on generalized Pöschl-Teller potential associated with dimer interaction.

Our future project, we will investigate many new results and applications on the generalized Pöschl-Teller potential with the dimer interaction.

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References

[1] A. Bhattacharjie and E. C. G. Sudarshan , *A class of solvable potentials*, Nuovo Cimento. **25**, 864–879, 1962.

[2] M. Simsek and Z. Yalcin,*Generalized Pöschl-Teller potential*, J. Math. Chem. **16**, 211–215, 1994.

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Interval uncertainty non-cooperative games

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The theory of non-cooperative games is well developed and widely represented in the scientific and educational literature (see for example [2, 4-10, 12]). The main results are related with the definition, existence, identification, finding and selection of preferred equilibrium situations. The classical concept of equilibrium [6] assumes that the payoffs of the players in each game situation are unambiguous. If the game allows polysemy of payoffs then the concept of equilibrium changes taking into account the nature of polysemy and the degree of awareness of the players. For example, in stochastic games, the polysemy of payoffs is due to the action of random factors. The law of distribution of these factors constitutes additional information known to the players. To study equilibrium situations, the apparatus of probability theory is used.

In games with ≪ nature ≫, the ambiguity of payoffs is associated with the uncertainty of nature's behavior. Due to the lack of additional information, this uncertainty is characterized as ≪ hopeless ≫. Equilibrium is understood in terms of guaranteed payoffs. Numerous game models with hopeless uncertainty arise in applications of economic and social sciences in which the choice of criteria and objective evaluations of strategies represent a complex independent problem.

One of the effective tools for studying mathematical models with uncertain factors is interval analysis. In this section, interval analysis is applied to non-cooperative games with hopeless payoff uncertainty. In other words, the model of the game assumes that the payoff of each participant can be any real number from some interval and there is no additional information about the distribution of payoffs within the interval. To determine equilibrium situations, partial ordering of intervals based on a numerical indicator is used. This allows us to reduce the original game to a new deterministic game [10] and generalize the classical theory [5, 6, 12].

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KEYWORDS: Interval indicator, Non-cooperative game, Equilibrium situation, Payoff uncertainty, Interval analysis

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References

[1] L. T. Ashchepkov and D. V. Davydov, *Interval inequality indicator: Properties and applications*, Computational Technologies **11 (4)**, 13–22, 2006.

[2] L. T. Ashchepkov, S. V. Gutorova, A. A. Karpachev and S. Li. *Interval matrix games*, Far Eastern Mathematical Journal **4 (2)**, 276–288, 2003.

[3] D. V. Dolgy, *On reduction of interval decision-making problems*, Adv. Stud. Contemp. Math. **33 (1)**, 33–49, 2023.

[4] J. McKinsey, *Introduction to game theory*, Moscow, Fizmatgiz, 1960.

[5] J. Neiman and O. Morgenstern, *Theory of games and economic behavior*, Moscow, Nauka, 1970.

[6] J. Nash, *Non-cooperative games*, Ann. Math. **54 (4)**, 286–295, 1951.

[7] G. Owen, *Game theory*, Moscow, Mir, 1971.

[8] L. A. Petrosyan, N. A. Zenkevich and E. A. Semina, *Game theory*, Higher school, Book House ≪University≫, Moscow, 1998.

[9] Russian translation: *Matrix games*, (Ed. by N.N. Vorobyov and M. Fizmatgiz), 205–221, 1961.

[10] V. N. Shashikhin, *Solution of an interval matrix game in mixed strategies*, Izvestiya of Russian Academy of Sciences, Theory and control systems, **5**, 97–104, 2001.

[11] A. Tsoukias and Ph. Vincke, *A characterization of PQI interval orders*, Discrete Appl. Math. **127 (2)**, 387–397, 2003.

[12] N. N. Vorobyov, *Fundamentals of game theory. Non-cooperative games*, Moscow, Nauka, 1984.

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On B-spline: Analysis of Apostol-Bernoulli numbers and Eulerian polynomials

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This study is inspired by the article of the second author [22], who calculated many explicit formulas and identities for a novel combining of generating functions with their functional and derivative equations for the certain family of special numbers and polynomials, the Bernstein basis functions, and also the uniform B-spline. Moreover, the aim of this presentation is to survey and investigate many properties of Eulerian numbers and polynomials with the aid of the Frobenius-Euler numbers and polynomials, the Apostol-Bernoulli numbers and polynomials. By using these relations we derive many new formulas for the B-splines and also the Bernstein basis functions.

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KEYWORDS: Generating functions, uniform B-splines, Bernstein basis functions, Apostol–Bernoulli numbers and polynomials, Eulerian numbers and polynomials

Introduction

The motivation of this presentation is to blend some certain family of the special polynomials, the (uniform) B-spline, and generating functions with their functional equations. The topics of this study have many applications in almost all areas of pure and applied mathematics and other sciences. For instance, in many scientific areas, the polynomial approach, which is among the approximation methods used for problem-solving, has been important applications. There are many relations among certain family of special polynomials, curves, and splines. We know that it may not always be possible to find the desired curve to solve problem, involving real-world problems or others. On the other hand, with the aid of (special) polynomials, which can be represented the desired curve and splines involving the Bezier curves, the Bernstein basis functions, the (uniform) B-spline etc. The other applications related to the numerical instabilities associated with the spline interpolation. These numerical instabilities can be overcome with the B-spline curve families. The B-spline curves are piecewise polynomials and form the basis for the entire family of spline curves. That is, all spline curves can be written as a linear combination of the B-spline (cf. [1]- [24]).

In order to derive novel formulas and relations of our work, we combine generating functions for the certain family of special numbers and polynomials, the Bernstein basis functions, and also the uniform B-spline, and others.

The following notations can be used throughout of this work:

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integers, the set of rational numbers, the set of real numbers, and the set of complex numbers. numbers, respectively.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Stirling numbers of the second kind are defined by

$$\frac{(e^t - 1)^m}{m!} = \sum_{n=0}^{\infty} S_2(n, m) \frac{t^n}{n!}, \tag{1}$$

where $t \in \mathbb{C}$ and $m \in \mathbb{N}_0$ (*cf.* [5, 19, 21, 24]).

The Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda)$ are defined by

$$K_n\left(t;\lambda\right)=\frac{t}{\lambda e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(\lambda)\frac{t^n}{n!}, \tag{2}$$

where λ is an arbitrary (real or complex) parameter and $|t|<2\pi$ when $\lambda=1$ and $|t|<|\log\lambda|$ when $\lambda\neq 1$ (*cf.* [2, 3, 19, 21, 24]).

Lemma 1. *Let $n \in \mathbb{N}$ with $n>1$. Then we have*

$$\mathcal{B}_n(\lambda)=\frac{n\lambda}{\left(\lambda-1\right)^n}\sum_{s=0}^{n-1}\left(-1\right)^s s!\lambda^{s-1}\left(\lambda-1\right)^{n-1-s}S_2\left(n-1,s\right) \tag{3}$$

(*cf.* [2]).

The Apostol-Bernoulli polynomials $\mathcal{B}_n(\omega;\lambda)$ are defined by

$$K_p\left(\omega,t;\lambda\right)=K_n\left(t;\lambda\right)e^{t\omega}=\sum_{n=0}^{\infty}\mathcal{B}_n(\omega;\lambda)\frac{t^n}{n!} \tag{4}$$

where $\lambda\neq 1$ (*cf.* [2]).

Substituting $\lambda=1$ into (4), we have the Bernoulli polynomials:

$$B_n\left(\omega\right)=\mathcal{B}_n\left(\omega;1\right)$$

when $\omega=0$, we also have $B_n=B_n(0)$ (*cf.* [2, 11, 21]).

Let ϕ be a complex number with $\phi\neq 1$. The Frobenius Euler polynomials $H_n(\omega;\phi)$ are defined by

$$F_P(t,\omega,\phi)=\frac{1-\phi}{e^t-\phi}e^{t\omega}=\sum_{n=0}^{\infty}H_n(\omega;\phi)\frac{t^n}{n!}, \tag{5}$$

where $|t|<2\pi$ when $\frac{1}{\phi}=1$ and $|t|<\left|\log\left(\frac{1}{\phi}\right)\right|$ when $\frac{1}{\phi}\neq 1$ (*cf.* [15, 21, 24]).

Substituting $\omega=0$ into (5), we have the Frobenius Euler numbers $H_n(\phi)$ are defined by

$$F_N(t,\phi)=\frac{1-\phi}{e^t-\phi}=\sum_{n=0}^{\infty}H_n(\phi)\frac{t^n}{n!}. \tag{6}$$

By using (6), we have

$$H_n(\phi)=\begin{cases}1 & \text{for } n=0\\ \frac{1}{\phi}\sum\limits_{j=0}^{n}\begin{pmatrix}n\\ j\end{pmatrix}H_j(\phi) & \text{for } n>0.\end{cases}$$

Combining (2) and (5) for $n>1$ yields

$$nH_{n-1}(\omega;\phi)=\frac{1-\phi}{\phi}\mathcal{B}_n\left(\frac{1}{\phi}\right). \tag{7}$$

The Eulerian polynomials are defined by

$$F_a(t,\lambda)=\frac{1}{1-\lambda e^{t(1-\lambda)}}=\sum_{n=0}^{\infty}\frac{A_n(\lambda)}{1-\lambda}\frac{t^n}{n!}, \tag{8}$$

where $\lambda \neq 1$, $A_n(\lambda)$ is a polynomial in λ of degree $n-1$ for $n>0$:

$$A_n(\lambda) = \sum_{j=0}^{n} A_{n,j}\lambda^j, \tag{9}$$

where $A_{n,j}$ is an integer number, known as the Eulerian numbers:

$$A_{n,j} = \sum_{v=0}^{j}(-1)^v \binom{n+1}{v}(j-v)^n, \tag{10}$$

$j = 1, 2, \ldots, n$, $0 \leq j < n$, $n \in \mathbb{N}$ (*cf.* [4, 5, 13]).

The Worpitzky identity for the Eulerian numbers is given as follows:

$$\omega^n = \sum_{v=0}^{n} \binom{\omega+v-1}{n} A_{n,v}, \tag{11}$$

where $\omega \in \mathbb{R}$, $n \in \mathbb{N}_0$ (*cf.* [4, 13]).

By using the umbral calculus method in (8), we get

$$A_n(\lambda) = \lambda \sum_{j=0}^{n} \binom{n}{j} (1-\lambda)^{n-j} A_j(\lambda),$$

where $A_0(\lambda) = 1$.

Generating functions for the Bernstein basis functions $B_j^k(\omega)$ are given by

$$f_{\mathbb{B},d}(t,\omega) = (\omega t)^d e^{t(1-\omega\omega)} = \sum_{k=d}^{\infty} d! B_d^k(\omega)\frac{t^k}{k!}, \tag{12}$$

where

$$B_d^k(\omega) = \binom{k}{d}\omega^d(1-\omega)^{k-d} \tag{13}$$

$0 \leq d \leq k$, and $d, k \in \mathbb{N}_0$, if $d > k$, then

$$B_d^k(\omega) = 0$$

(*cf.* [23, 20]).

In recent years, many studies have been carried out covering the generating functions of Bernstein base functions and their applications in many different scientific areas (*cf.* [1, 7, 9, 10, 16, 17, 14, 23]).

Goldman [6] gave the following generating function for the uniform B-splines from $N_{0,n}(\omega; p)$:

$$\begin{aligned} G_0(\omega,t;p) &= \sum_{j=0}^{p}(-1)^j\left(\frac{(\omega-j)^j t^j}{j!} + \frac{(\omega-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{(\omega-j)t} \\ &= \sum_{n=0}^{\infty} N_{0,n}(\omega;p)t^n, \end{aligned} \tag{14}$$

$p \leq \omega \leq p+1$.

By using (14), Goldman [6, Theorem 3] gave the following well-known Schoenberg's identity and the de Boor recurrence for the uniform B-splines, respectively:

$$N_{0,n}(\omega;p) = \frac{1}{n!}\sum_{j=0}^{p}(-1)^j\binom{n+1}{j}(\omega-j)^n, \tag{15}$$

where $p \leq \omega \leq p+1$ and

$$N_{0,n}(\omega;p) = \frac{\omega}{n} N_{0,n-1}(\omega;p) + \frac{n+1-\omega}{n} N_{1,n-1}(\omega;p)$$

(*cf.* [6, Theorem 3]).

On the other hand, the second author [22] generalized both the Schoenberg's identity and the de Boor recurrence for the uniform B-splines. He [22] also gave the following novel formulas:

$$\mathcal{B}_{m+1}(\lambda) = (-1)^m (\lambda-1)^{m+1} (m+1)! \sum_{j=0}^{m} N_{0,m}(j;j)\lambda^j \tag{16}$$

and

$$N_{0,m}(p;p) = \frac{1}{m!} A_m(p), \tag{17}$$

where $m \in \mathbb{N}_0$

$$A_{m-1}(\lambda) = -\frac{(1-\lambda)^m}{m\lambda} \mathcal{B}_m(\lambda) \tag{18}$$

and

$$A_m(\lambda) = \sum_{n=1}^{m} n!(1-\lambda)^{m-n}\lambda^n S_2(m,n) \tag{19}$$

where $m \in \mathbb{N}$ (*cf.* [22]).

Boyadzhiev [3] gave the following relation between the Apostol-Bernoulli numbers and the geometric polynomials $W_n(w)$:

$$\mathcal{B}_n(\lambda) = \frac{n}{\lambda-1} W_{n-1}\left(\frac{\lambda}{1-\lambda}\right), \tag{20}$$

where $n \in \mathbb{N}$ and

$$W_n(\lambda) = \sum_{j=0}^{n} j! S_2(n,j)\lambda^j. \tag{21}$$

Relations among the uniform B-spline, Apostol-Bernoulli numbers, Frobenius Euler numbers Eulerian numbers and the Bernstein Basis Functions

In this section, we give new formulas including uniform B-splines, Apostol-Bernoulli numbers, Frobenius Euler polynomials and numbers, and Eulerian numbers. We also give some functional equations to generating functions for uniform B-splines and Bernstein basis functions. With these equations, we work on series representations for uniform B-splines and Bernstein basis functions.

By combining (3) with (18), we arrive at the following theorem:

Theorem 2. *Let $n \in \mathbb{N}_0$*

$$A_n(\lambda) = (-1)^n \sum_{s=0}^{n} (-1)^s s! \lambda^{s-1} (\lambda-1)^{n-s} S_2(n,s).$$

By combining (18) with (20), we have

$$A_n(\lambda) = \frac{(1-\lambda)^n}{\lambda} W_n\left(\frac{\lambda}{1-\lambda}\right).$$

By using the above equation with (21), we obtain

$$A_n(\lambda) = \sum_{j=0}^{n} j!\,(1-\lambda)^{n-j}\,\lambda^{j-1} S_2(n,j).$$

By combining the above equation with (13), we arrive at the following theorem:

Theorem 3. *Let $n \in \mathbb{N}_0$*

$$A_n(\lambda) = \sum_{j=0}^{n} j! \frac{S_2(n,j)}{\binom{n+j+1}{j}} B_{j-1}^{n-1}(\lambda).$$

By combining (18) with (16), we arrive at the following theorem:

Theorem 4. *Let $n \in \mathbb{N}_0$*

$$A_n(\lambda) = \frac{(\lambda-1)^{2n+2}}{\lambda} n! \sum_{j=0}^{n} N_{0,n}(j;j)\lambda^j. \tag{22}$$

Conclusion

This work was inspired by the article of the second author [22], who calculated many explicit formulas and identities for a novel combining of generating functions with their functional and derivative equations for the certain family of special numbers and polynomials, the Bernstein basis functions, and also the uniform B-spline. We investigated various properties of Eulerian numbers and polynomials by using the Frobenius-Euler numbers and polynomials, the Apostol-Bernoulli numbers and polynomials. By using these relations, we derived some novel formulas for the B-splines and also the Bernstein basis functions.

Our future project will be to investigate applications of the uniform B-spline associated with splines, and other special numbers and polynomials.

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References

[1] M. Acikgoz and S. Araci, *On generating function of the Bernstein polynomials*, AIP Conf. Proc. **1281**, 1141–1143, 2010.

[2] T. M. Apostol, *On the Lerch zeta function*, Pac. J. Math. **1**, 161–167, 1951.

[3] K. N. Boyadzhiev, *Apostol-Bernoulli functions, derivative polynomials and Eulerian polynomials*, https://arxiv.org/ftp/arxiv/papers/0710/0710.1124.pdf.

[4] L. Carlitz, D. P. Roselle and R. Scoville, *Permutations and sequences with repetitions by number of increase*, J. Combin. Theory **1**, 350–374, 1966.

[5] L. Comtet, *Advanced combinatorics*, D. Reidel Publication Company, Dordrecht-Holland/Boston-U.S.A., 1974.

[6] R. Goldman, *Generating functions for uniform B-splines*, In: Mathematical Methods for Curves and Surfaces, Lecture Notes in Computer Science, Floater, Vol. 8177, Springer, Berlin, Heidelberg, 2014.

[7] R. Goldman, P. Simeonov and Y. Simsek, *Generating functions for the q-Bernstein bases*, Siam J. Discrete Math. **28 (3)**, 1009–1025, 2014.

[8] D. Gun and Y. Simsek, *Modification exponential euler type splines derived from Apostol-Euler numbers and polynomials of complex order*, Appl. Anal. Discrete Math. **17**, 197–215, 2003.

[9] I. Kucukoglu, B. Simsek and Y. Simsek, *Multidimensional Bernstein polynomials and Bezier curves: Analysis of machine learning algorithm for facial expression recognition based on curvature*, Appl. Math. Comput. **344**, 150–162, 2019.

[10] G. G. Lorentz, *Bernstein polynomials*, Chelsea Publishing Company, New York, 1986.

[11] Q. M. Luo and H. M. Srivastava, *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl. **308**, 290–302, 2005.

[12] T. K. Petersen, *Eulerian numbers*, Birkhäuser, Basel, 2015.

[13] P. L. Putzer and M. Hauss, *Eulerian numbers with fractional order parameters*, Aequationes Math. **46**, 119–142, 1993.

[14] D. Salomon, *Curves and Surfaces for Computer Graphics*, Springer-Verlag, New York, 2006.

[15] I. J. Schoenberg, *Cardinal interpolation and spline functions IV. The exponential Euler splines*, In: Linear Operators and Approximation Lineare Operatoren und Approximation, Birkhäuser, Basel, 1972.

[16] B. Simsek, *Formulas derived from moment generating functions and Bernstein polynomials*, Appl. Anal. Discrete Math. **13 (3)**, 839–848, 2019.

[17] B. Simsek, *A note on characteristic function for Bernstein polynomials involving special numbers and polynomials*, Filomat **34 (2)**, 543–549, 2020.

[18] Y. Simsek, *Interpolation functions of the Eulerian type polynomials and numbers*, Advanced Studies Contemp. Math. **23 (2)**, 301–307, 2013.

[19] Y. Simsek, *Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications*, Fixed Point Theory Appl. **2013**, 1–28, 2013.

[20] Y. Simsek, *Functional equations from generating functions: A novel approach to deriving identities for the Bernstein basis functions*, Fixed Point Theory Appl. **2013**, 1–13, 2013.

[21] Y. Simsek, *Explicit formulas for p-adic integral: Approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 1–76, 2019.

[22] Y. Simsek, *Novel formulas for B-Splines, Bernstein basis functions, and special numbers: Approach to derivative and functional equations of generating functions*, Mathematics **12**, 2024; Article ID: 65, https://doi.org/10.3390/math12010065.

[23] Y. Simsek and M. Acikgoz, *A new generating function of (q-) Bernstein-type polynomials and their interpolation function*, Abstr. Appl. Anal. **2010**, Article ID: 769095.

[24] H. M. Srivastava and J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

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On generating functions of Jacobi and Bernstein polynomials

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The aim of this study is to establish the Jacobi polynomials in terms of Bernstein basis functions. First of all, the definition and some important properties of the Jacobi polynomials are given. As the main part of the study, we construct generating functions of these Jacobi polynomials written as linear combinations of Bernstein basis functions.

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KEYWORDS: Bernstein polynomials, Jacobi polynomials, Generating functions

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References

[1] G. Pittaluga, L. Sacripante and H. M. Srivastava, *Some families of generating functions for the Jacobi and related orthogonal polynomials*, J. Math. Anal. Appl. **238 (2)**, 385–417, 1999.

[2] E. D. Rainville, *Special functions*, The Macmillan Company, New York, 1960.

[3] Y. Simsek, *Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications*, Fixed Point Theory Appl. **2013**, 2013; Article ID: 87.

[4] Y. Simsek, *Analysis of the Bernstein basis functions: an approach to combinatorial sums involving binomial coefficients and Catalan numbers*, Math. Methods Appl. Sci. **38 (14)**, 3007–3021, 2015.

[5] H. M. Srivastava and M.-P. Chen, *Orthogonality relations and generating functions for Jacobi polynomials and related hypergeometric functions*, Appl. Math. Comput. **68 (2-3)**, 153–188, 1995.

[6] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, 1964.

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On the special cases of twice-iterated big q-Appell polynomials

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In this paper, we choose some special cases of the determining functions $a_1(t)$ and $a_2(t)$ of the twice-iterated Big q-Appell Polynomials. Therefore, we obtain some twice-iterated polynomials. We present the recurrence relations, shift operator and difference equations which satisfy by the twice iterated big q-Bernoulli-Euler polynomials, twice-iterated big q-Bernoulli polynomials, twice-iterated big q-Euler polynomials, twice-iterated big q-Genocchi polynomials.

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References

[1] W. A. Al-Salam,*q-Appell polynomials*, Annali di Matematica **77**, 31–45, 1967.

[2] N. Biricik, B. Çekim and M. A. Özarslan, *Sequences of twice-iterated* $\Delta\omega$*-Gould–Hopper Appell polynomials*, Journal of Taibah University for Science **18 (1)**, 2286–714, 2024.

[3] M. E. Keleshteri and N. I. Mahmudov, *A study on q-Appell polynomials from determinantal point of view*, Appl. Math. Comput. **260**, 351–369, 2015.

[4] S. Khan and M. Riyasat, *Determinant approach to the 2-Iterated q-Appell and mixed type q-Appell polynomials*, 2016; ArXiv:1606.04265.

[5] P. N. Sadjang, *On* (p, q)*-Appell polynomials*, Anal. Math. **45 (3)**, 583–598, 2019.

[6] A. Sharma and A. M. Chak, *The basic analogue of a class of polynomials*, Riv. Math. Univ. Parma (N.S.) **5**, 325–337, 1954.

[7] H. M. Srivastava, S. Khan and M. Riyasat, *q-Difference equations for the 2-iterated q-Appell and mixed type q-Appell polynomials*, Arab. J. Math. **8 (1)**, 63–77, 2019.

[8] H. M. Srivastava, M. A. Özarslan and B. Y. Yaşar, *Difference equations for a class of twice-iterated* Δ_h*-Appell sequences of polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113 (3)**, 1851–1871, 2019.

[9] H. M. Srivastava, B. Y. Yaşar and M. A. Özarslan, *A class of big (p, q)-Appell polynomials and their associated difference equations*, Filomat **33 (10)**, 3085–3121, 2019.

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Convolution sums of restricted divisor functions including a pairwise coprime conditions

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The study of general convolution sums, $\sum_{i=1}^{N-1} g(i)g(N-i)$, has been prevalent for a long time, but the form of $\sum_{\substack{i=1\\ \gcd(i,N-i)=1}}^{N-1} g(i)g(N-i)$ is not well-known. The aim of this article is to find formulas $\overline{\mathcal{A}}$ and $\overline{\mathcal{K}}$ using the Dirichlet convolution sums.

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KEYWORDS: Dirichlet convolution, Restricted divisor functions

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This paper is dedicated to Professor Taekyun KIM on the occasion of his 60th anniversary.

References

[1] B. Cho, *Convolution sums of divisor functions for prime levels*, Int. J. Number Theory **16**, 537–546, 2020.

[2] N. Cheng and K. S. Williams, *Convolution sums involving the divisor function*, Proc. Edinb. Math. Soc. **47 (3)**, 561–572, 2004.

[3] H. M. Farkas, *On an arithmetical function II*, Contemp. Math. **382**, 121–130, 2005.

[4] A. Kim, D. Kim and Y. Li, *Convolution sums arising from divisor functions*, J. Korean Math. Soc. **50 (2)**, 331–360, 2013.

[5] P. J. McCarthy, *Introduction to arithmetical functions*, Springer Science & Business Media, New York, 2012.

[6] S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22**, 159–184, 1916.

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A note of iterations for matrix

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First of all, congratulations on Professor Kim Tae-gyun's 60th birthday. This paper analyzes repeated results in a matrix. The facts used are the Cayley-Hamilton theorem of linear algebra and the theory of elliptic curves. Of course, we define matrix sequences and investigate their iterated properties.

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References

[1] D. Kim and A. Bayad, *Polygon numbers associated with the sum of odd divisors function*, Exp. Math. **26**, 287–297, 2017.

[2] D. Kim, A. Bayad and S. Park, *Iterated stable numbers and iterated amicable pairs,* Math. Methods Appl. Sci. **44**, 12479–12499, 2021.

[3] S. Lang, *Linear algebra*, Springer-Velag, New York, 1987.

[4] J. H. Silvermann, *The arithmetic of Elliptic curves*, Springer-Verlag, New York, 1992.

[5] L. C. Washington, *Elliptic curves number theory and cryptography*, CRC Press, 2008.

[6] F. Zhang, *Matrix theory*, Springer, New York, 1999.

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Exponential sums for classical groups and their applications

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The aim of this survey paper is to review Gauss sums for classical groups over finite fields and show some of their applications. The explicit computations of Gauss sums for such groups were done in three different ways. Here, in the case of the Gauss sum for the symplectic group $Sp(2n, q)$, we give a sketch of proof only for the method which uses the Bruhat decomposition with respect to a maximal parabolic subgroup of $Sp(2n, q)$. Then we present three applications of Gauss sums for classical groups over finite fields to the evaluations of Hodges' Kloosterman sums, counting the number of elements in classical groups over finite fields with given traces, and constructions of linear codes and power moments of Kloosterman sums.

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KEYWORDS: Exponential sum, Classical group, Bruhat decomposition, Maximal parabolic subgroup, Hodges' Kloosterman sums, Linear codes, Power moments of Kloosterman sums

Introduction

The present article is a survey paper and an enlarged version of the earlier one (*cf.* [13]) about twenty eight years ago. The purpose of this paper is to consider 'Gauss sums' for classical groups over finite fields and show how explicitly they can be evaluated, and to give three applications of them.

The actual computations about explicit expressions of such Gauss sums were done by using three different ways. The first method uses Bruhat decompositions with respect to suitable maximal parabolic subgroups of the classical groups, the second one expresses the Gauss sums as the alternating sum of traces of Frobenius maps acting on Weyl group invariant part of certain cohomology groups of $(G/T)T$, where T is a maximal torus of G, and the third one utilizes some consequences of Deligne-Lusztig character theory of G and structure theory on maximal tori of G. Here we are content with giving a sketch of the proof by using the first method for the Gauss sum associated with symplectic groups over finite fields.

Astonishingly enough, these methods have turned out to be very fruitful. We relate the Gauss sums for classical groups to Hodges' Kloosterman sums over finite fields (*cf.* [1], [8]- [11]). Also, it is shown that the number of elements in the classical groups over finite fields with the given trace values can be obtained from our explicit expressions for the corresponding Gauss sums. In addition, we construct binary and ternary linear codes associated with classical groups of small order and obtain recursive formulas for (complete or incomplete) power moments of Kloosterman sums in terms of frequencies of weights in those codes. These are done by using the explicit expressions of Gauss sums for the classical groups over finite fields and the Pless power moment identity. Furthermore, we can construct many infinite families of linear codes from double cosets with respect to certain maximal parabolic subgroups of the classical groups over finite fields and get recursive formulas for power moments of Kloosterman sums. These are done via Pless power moment identity and by utilizing the explicit expressions of exponential sums over those double cosets related to the evaluations of Gauss sums for classical groups.

Classical sums

The following notations will be used throughout this paper (*cf.* [45]).

$q = p^r$ (p a prime, $r \in \mathbb{Z}_{>0}$),

$\mathbb{F}_q$=the finite field with q elements,

$tr(x) = tr_{\mathbb{F}_q/\mathbb{F}_p}(x) = x + x^p + \cdots + x^{p^{r-1}}$ the trace function $\mathbb{F}_q \to \mathbb{F}_p$,

$\lambda(x) = e^{2\pi i tr(x)/p}$ the canonical additive character of $\mathbb{F}_q$,

TrA = the trace of A for a square matrix A,

${}^tB$ = the transpose of B for any matrix B.

Let ψ be an additive character of $\mathbb{F}_q$ (i.e., $\psi \in$ Hom $(\mathbb{F}_q^+, \mathbb{C}^\times)$). Then it is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q$, so that

$$\psi(x) = \lambda(ax) = \exp\left\{\frac{2\pi i}{p}(ax + (ax)^p + \cdots + (ax)^{p^{r-1}})\right\}.$$

Furthermore, $\psi(x) = \lambda(ax)$ is nontrivial if $a \neq 0$.

Let χ be a multiplicative character of $\mathbb{F}_q$ (i.e., $\chi \in$ Hom$(\mathbb{F}_q^\times, \mathbb{C}^\times)$). Then the (classical) Gauss sum is defined as :

$$G(\chi, \psi) = \sum_{\alpha \in \mathbb{F}_q^\times} \chi(\alpha)\psi(\alpha).$$

For a nontrivial additive character ψ and $a \in \mathbb{F}_q^\times$, the (classical) Kloosterman sum $K(\psi; a)$ is defined by

$$K(\psi; a) = \sum_{\alpha \in \mathbb{F}_q^\times} \psi(\alpha + a\alpha^{-1}).$$

The Kloosterman sum was introduced in 1926 (*cf.* [43]) to give an estimate for the Fourier coefficients of modular forms.

For any positive integer h, the hth moment of Kloosterman sums is defined as

$$MK^h = \sum_{\alpha \in \mathbb{F}_q^\times} K(\lambda; \alpha)^h.$$

Explicit computations on power moments of Kloosterman sums were initiated in the paper [49] of Salié in 1931, where it is shown that for any odd prime power q,

$$MK^h = q^2 M_{h-1} - (q-1)^{h-1} + 2(-1)^{h-1}, \quad (h \geq 1).$$

Here $M_0 = 0$, and for $h \in \mathbb{Z}_{>0}$,

$$M_h = \mid \{(\alpha_1, \ldots, \alpha_h) \in (\mathbb{F}_q^*)^h \mid \sum_{j=1}^{h} \alpha_j = 1 = \sum_{j=1}^{h} \alpha_j^{-1}\} \mid .$$

For $q = p$ odd prime, Salié obtained MK^1, MK^2, MK^3, MK^4 in [49] by determining M_1, M_2, M_3. Assume now that $q = 3^r$. Recently, Moisio was able to find explicit expressions of MK^h, for $h \leq 10$ (cf. [46]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the ternary Melas code of length $q-1$, which were known by the work of Geer, Schoof and Vlugt in [7].

For a nontrivial additive character ψ of $\mathbb{F}_q$, and $a \in \mathbb{F}_q^{\times}$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ is defined as :

$$K_{GL(t,q)}(\psi; a) = \sum_{g \in GL(t,q)} \psi(\operatorname{Tr} g + a \operatorname{Tr} g^{-1}),$$

where $GL(t,q)$ is the general linear group consisting of all $t \times t$ invertible matrices with entries in $\mathbb{F}_q$. In [14], it is shown that $K_{GL(t,q)}(\psi; a)$ satisfies the following recurrence relation.

Theorem 1. *For integers $t \geq 2$, we have*

$$K_{GL(t,q)}(\psi; a) = q^{t-1} K_{GL(t-1,q)}(\psi; a) K(\psi; a) + q^{2t-2}(q^{t-1} - 1) K_{GL(t-2,q)}(\psi; a),$$

where we understand that $K_{GL(0,q)}(\psi; a) = 1$.

From Theorem 1, we can derive the following expression given by

$$K_{GL(t,q)}(\psi; a) = q^{\frac{1}{2}(t-2)(t+1)} \sum_{l=1}^{[(t+2)/2]} q^l K(\psi; a)^{t+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1), \qquad (1)$$

where the inner sum is over all integers $j_1, \cdots, j_{l-1}$ satisfying $2l - 1 \leqslant j_{l-1} \leqslant j_{l-2} \leqslant \cdots \leqslant j_1 \leqslant t+1$ (with the understanding that the inner sum is 1 for $l = 1$).

Classical groups

Let $J_{2n}, J_{2n+1}, J_{2n}^{+}, J_{2n}^{-}$ be the matrices respectively given by

$$\begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1_{n-1} & 0 & 0 \\ 1_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\epsilon \end{bmatrix},$$

with a fixed element $\epsilon \in \mathbb{F}_q^{\times} \backslash \mathbb{F}_q^{\times 2}$. Then our classical groups are as follows.

$$GL(n,q) = \{g | g \,\text{is of size}\, n \times n \,\text{with entries in}\, \mathbb{F}_q, \det g \neq 0\}.$$

$$GSp(2n,q) = \{g \in GL(2n,q) | \, {}^t g J_{2n} g = \nu(g) J_{2n}, \,\text{for some}\, \nu(g) \in \mathbb{F}_q^{\times}\}.$$

$$O(2n+1,q) = \{g \in GL(2n+1,q) | {}^t g J_{2n+1} g = J_{2n+1}\}.$$

$$O^{+}(2n,q) = \{g \in GL(2n,q) | {}^t g J_{2n}^{+} g = J_{2n}^{+}\}.$$

$$O^{-}(2n,q) = \{g \in GL(2n,q) | {}^t g J_{2n}^{-} g = J_{2n}^{-}\}.$$

For $O(2n+1,q), O^+(2n,q)$, *and* $O^-(2n,q)$, we assume that $\operatorname{char}\mathbb{F}_q \neq 2$.

$$U(2n,q^2) = \{g \in GL(2n,q^2)|\, {}^*gJ_{2n}^+g = J_{2n}^+\}.$$

$$U(2n+1,q^2) = \{g \in GL(2n+1,q^2)|{}^*gJ_{2n+1}g = J_{2n+1}\}.$$

Here $\tau : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ is the Frobenius automorphism given by $\tau\alpha = \alpha^\tau = \alpha^q \ {}^*g = {}^tg^\tau$, and for $A = (a_{ij}), A^\tau = (a_{ij}^\tau)$.
Except for

$$Sp(2n,q) = \{g \in GL(2n,q)|{}^tgJ_{2n}g = J_{2n}\},$$

$SL(n,q), SO(2n+1,q), SO^+(2n,q), SO^-(2n,q), SU(2n,q^2), SU(2n+1,q^2)$ are the corresponding ones intersected with $\{\det = 1\}$.

Assume that char $\mathbb{F}_q = 2$. In this case, the group $O(2n+1,q)$ is defined as the group of all isometries of the nondegenerate quadratic space $(\mathbb{F}_q^{(2n+1)\times 1}, \theta)$, where θ is the nondegenerate quadratic form on the vector space $\mathbb{F}_q^{(2n+1)\times 1}$ of all $(2n+1)\times 1$ column vectors over $\mathbb{F}_q$. These groups can be given explicitly in terms of invertible matrices. Analogously to this, $O^+(2n,q)$ and $O^-(2n,q)$ are defined. For details on these, one refers to [41].

Main results

Gauss sums for classical groups

Let χ be a multiplicative character of $\mathbb{F}_q$, and let ψ be a nontrivial additive character of $\mathbb{F}_q$. Then we consider, respectively called the Gauss sum for G and that for SG,

$$\sum_{g\in G} \chi(\det g)\psi(\operatorname{Tr} g),$$

and

$$\sum_{g\in SG} \psi(\operatorname{Tr} g),$$

where $G = GL(n,q), GSp(2n,q), O(2n+1,q), O^+(2n,q), O^-(2n,q)$, and $SG = SL(n,q)$, $Sp(2n,q), SO(2n+1,q), SO^+(2n,q), SO^-(2n,q)$.

Let χ' be a multiplicative character of $\mathbb{F}_{q^2}, \psi' = \psi \circ \operatorname{tr}_{\mathbb{F}_q^2/\mathbb{F}_q}$ (with ψ a nontrivial additive character of $\mathbb{F}_q$ as above). Then we also consider

$$\sum_{g\in G} \chi'(\det g)\psi'(\operatorname{Tr} g),$$

and

$$\sum_{g\in SG} \psi'(\operatorname{Tr} g),$$

where $G = U(2n,q^2), U(2n+1,q^2)$ and $SG = SU(2n,q^2), SU(2n+1,q^2)$.

Our concern is to find explicit expressions for these sums. Here we briefly go over the Gauss sum for $Sp(2n,q)$. As to the Gauss sums for the other classical groups over finite fields, the reader refers to the papers [15]- [24], [40] and [41]. All of these were done in the same spirit as for the symplectic group by using Bruhat decompositions with respect to suitable maximal parabolic subgroups in the classical groups over finite fields. Many of these results were mentioned in [44] and [47].

As we remarked earlier, these computations were also carried out by using two completely different methods. For one, the Gauss sum $\sum_{g\in G}\psi(Trg)$, for $G = GL(n,q), SL(n,q), Sp(2n,q), SO^{+}(2n,q), SO(2n+1,q)$, was expressed as the alternating sum of traces of Frobenius maps acting on Weyl group invariant part of certain cohomology groups of $(G/T)T$, where T is a maximal torus of G. For these, one refers to [2], [5] and [50]. For another, the Gauss sum $\sum_{g\in G}\chi(detg)\psi(Trg)$ for the classical group G was calculated by using some consequences of Deligne-Lusztig character theory of G and structure theory on maximal tori of G (*cf.* [6]). Indeed, these calculations were done for $GL(n,q), SL(n,q), U(n,g^2), SU(n,q^2)$ in [3] and for $Sp(2n,q), GSp(2n,q), SO^{+}(2n,q), SO^{-}(2n,q), SO(2n+1,q)$ in [4].

Let $P = P(2n,q)$ be the maximal parabolic subgroup of $Sp(2n,q)$ defined by:

$$P(2n,q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^{t}A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \mid A \in GL(n,q), {}^{t}B = B \right\}.$$

Then, with respect to $P = P(2n,q)$, the Bruhat decomposition of $Sp(2n,q)$ is given by

$$Sp(2n,q) = \coprod_{r=0}^{n} P\sigma_r P,$$

where

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ -1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in Sp(2n,q).$$

Put, for each r with $0 \le r \le n$,

$$A_r = \{g \in P(2n,q) \mid \sigma_r g \sigma_r^{-1} \in P(2n,q)\}.$$

Expressing $Sp(2n,q)$ as a disjoint union of right cosets of $P = P(2n,q)$, the Bruhat decomposition in (12) can be written as

$$Sp(2n,q) = \coprod_{r=0}^{n} P\sigma_r (A_r \setminus P). \tag{2}$$

The order of the general linear group $GL(n,q)$ is given by

$$g_n = \prod_{j=0}^{n-1}(q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n}(q^j - 1). \tag{3}$$

For integers n, r with $0 \le r \le n$, the q-binomial coefficients are defined as:

$$\left[{n \atop r} \right]_q = \prod_{j=0}^{r-1}(q^{n-j} - 1)/(q^{r-j} - 1). \tag{4}$$

In [14], it is shown that

$$|A_r| = g_r g_{n-r} q^{\binom{n+1}{2}} q^{r(2n-3r-1)/2}. \tag{5}$$

Also, it is immediate to see that

$$|P(2n,q)| = q^{\binom{n+1}{2}} g_n. \tag{6}$$

So, from (3)-(6), we get

$$|A_r \setminus P(2n,q)| = q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q ,$$

and

$$|P(2n,q)\sigma_r P(2n,q)| = |P(2n,q)|^2 |A_r|^{-1} = q^{n^2} \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\binom{r}{2}} q^r \prod_{j=1}^{n} (q^j - 1). \qquad (7)$$

Also, from (2) and (7), we have

$$|Sp(2n,q)| = \sum_{r=0}^{n} |P(2n,q)|^2 |A_r|^{-1} = q^{n^2} \prod_{j=1}^{n} (q^{2j} - 1),$$

where one can apply the following q-binomial theorem with $x = -q$:

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}} x^r = (x;q)_n,$$

with $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$.

Now, it is shown in [14] that the Gauss sum for $Sp(2n,q)$ is given by:

$$\sum_{g \in Sp(2n,q)} \psi(Trg) = \sum_{r=0}^{n} \sum_{g \in P\sigma_r P} \psi(Trg),$$

where

$$\begin{aligned} &\sum_{g \in P\sigma_r P} \psi(Trg) = |A_r \setminus P| \sum_{g \in P} \psi(Trg\sigma_r) \\ &= q^{\binom{n+1}{2}} |A_r \setminus P| q^{r(n-r)} a_r K_{GL(n-r,q)}(\psi;1) \\ &= \begin{cases} 0, & \text{if } r \text{ is odd,} \\ q^{\binom{n+1}{2}} q^{rn - \frac{1}{4}r^2} \begin{bmatrix} n \\ r \end{bmatrix}_q \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-r,\ q)}(\psi;1), & \text{if } r \text{ is even.} \end{cases} \end{aligned}$$

Here $a_0 = 1$, and, for $r \in \mathbb{Z}_{>0}$, a_r denotes the number of all $r \times r$ nonsingular alternating matrices over $\mathbb{F}_q$, which is given by

$$a_r = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ q^{\frac{r}{2}(\frac{r}{2}-1)} \prod_{j=1}^{\frac{r}{2}} (q^{2j-1} - 1), & \text{if } r \text{ is even.} \end{cases}$$

(cf. [14], Proposition 5.1).

Thus the Gauss sum for $Sp(2n,q)$ is given by

$$\begin{aligned} \sum_{g \in Sp(2n,q)} \psi(Trg) = q^{\binom{n+1}{2}} &\sum_{0 \le r \le n,\ r \text{ even}} q^{rn - \frac{1}{4}r^2} \begin{bmatrix} n \\ r \end{bmatrix}_q \\ &\times \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-r,\ q)}(\psi;1). \end{aligned} \qquad (8)$$

By combining (1) with (8), we finally obtain the following result.

Theorem 2. *For any nontrivial additive character ψ of $\mathbb{F}_q$, we have*

$$\sum_{g\in Sp(2n,q)} \psi(Trg) = q^{n^2-1}\sum_{r=0}^{[\frac{n}{2}]} q^{r(r+1)} \left[{n \atop 2r}\right]_q \prod_{j=1}^{r}(q^{2j-1}-1)$$
$$\times \sum_{l=1}^{[\frac{n-2r+2}{2}]} q^l K(\psi;1)^{n-2r+2-2l} \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),$$

where the unspecified sum runs over the set of integers $j_1,\dots,j_{l-1}$ satisfying $2l-1 \leq j_{l-1} \leq \cdots \leq j_1 \leq n-2r+1$.

Applications to Hodges' Kloosterman sums

Assume that $q=p^d$, with $p>2$.

The Hodges' Kloosterman sum over nonsingular alternating matrices, for $2t\times 2t$ alternating matrices A,B over $\mathbb{F}_q$, is defined as :

$$K_{alt,2t}(A,B)=\sum_g \lambda(\mathrm{tr}\,(Ag+Bg^{-1})),$$

where the sum is over all nonsingular alternating matrices g of size $2t$.
From Theorem 6 of [10], and for $a\in\mathbb{F}_q^\times$,

$$\begin{aligned} K_{alt,2t}(\frac{a^2}{4}C^{-1},C) &= K_{alt,2t}(\frac{a^2}{4}J_{2n}^{-1},J_{2n}) \\ &= q^{-n}\sum_{g\in Sp(2n,q)}\lambda(a\,\mathrm{tr}\,g), \end{aligned} \tag{9}$$

where C is any nonsingular alternating matrix of size $2t$.

We note here that some of general properties of the Hodges' Kloosterman sum were investigated, and, for A or B zero, it was evaluated in [10]. However, it has never been explicitly evaluated for both A and B nonzero. From the identity in (9) and the explicit expression of $\sum_{g\in Sp(2n,q)}\lambda(a\,\mathrm{tr}\,g)$, we get an expression for $K_{alt,2t}(\frac{a^2}{4}C^{-1},C)$. For the connection of similar nature between Gauss sums for classical groups over finite fields and various weighted partitions of Hodges, one refers to [1], [8], [9] and [11]. In [49], H. Saito reconsidered, without knowing the existence of Hodges' papers [8]-[11], certain generalizations of Hodges' Kloosterman sums, but with only one nonzero argument and determined their explicit values. His purpose was to apply those to twisting operators acting on Siegel modular forms and zeta functions associated with the prehomogeneous vector space of symmetric matrices.

In [25], two exponential sums, one generalizing Hodges' Kloosterman sums and the other generalizing Saito's sums to the ones with two arguments, were suggested.

Applications to counting the number of elements in classical groups with given traces

Here we apply our results in Section 2.1 to the problem of counting the number of elements in classical groups with given matrix trace values. Here we treat only the case for the symplectic group $Sp(2n,q)$. For the other classical groups over finite fields, one refers to [20]- [24].

If $G(q)$ is one of the finite classical groups over $\mathbb{F}_q$, then, for each $\beta \in \mathbb{F}_q$, we put

$$N_{G(q)}(\beta) = |\{g \in G(q) \,|\, \mathrm{Tr}\, g = \beta\}|.$$

For ψ a nontrivial additive character of $\mathbb{F}_q$, we have

$$qN_{G(q)}(\beta) = |G(q)| + \sum_{\alpha \in \mathbb{F}_q^{\times}} \psi(-\beta\alpha) \sum_{g \in G(q)} \psi(\alpha\, \mathrm{Tr}\, g). \tag{10}$$

Lemma 3. *Let ψ be a nontrivial additive character of $\mathbb{F}_q$, $\beta \in \mathbb{F}_q$, and let m be a nonnegative integer. Then*

$$\sum_{\alpha \in \mathbb{F}_q^{\times}} \psi(-\beta\alpha) K(\psi; \alpha^2)^m = q\delta(m, q; \beta) - (q-1)^m,$$

where, for $m \geqslant 1$,

$$\delta(m, q; \beta) = |\{(\alpha_1, \cdots, \alpha_m) \in (\mathbb{F}_q^{\times})^m \,|\, \alpha_1 + \alpha_1^{-1} + \cdots + \alpha_m + \alpha_m^{-1} = \beta\}| \tag{11}$$

and

$$\delta(0, q; \beta) = \begin{cases} 1, & \text{if } \beta = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Now, applying (10) to $G(q) = Sp(2n, q)$, and using Theorem 2 and Lemma 3, we get the following result.

Theorem 4. *For each $\beta \in \mathbb{F}_q$, the number of elements $N_{Sp(2n,q)}(\beta)$ of $g \in Sp(2n, q)$ with $Tr\, g = \beta$ is given by*

$$q^{n^2-1} \prod_{j=1}^{n} (q^{2j} - 1)$$

plus

$$\begin{aligned} &q^{n^2-1} \sum_{r=0}^{[n/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r \end{bmatrix}_q \prod_{j=1}^{r} (q^{2j-1} - 1) \\ &\times \sum_{l=1}^{[(n-2r+2)/2]} q^l \left(\delta(n - 2r + 2 - 2l, q; \beta) - q^{-1}(q-1)^{n-2r+2-2l} \right) \\ &\times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1), \end{aligned}$$

where the innermost sum runs over the same set of integers as in Theorem 2, and $\delta(m, q; \beta)$ is as in (11) *and* (12).

Applications to constructions of linear codes and power moments of Kloosterman sums

Here we construct two ternary linear codes associated with the symplectic groups $Sp(2, q)$ and $Sp(4, q)$. Here $q = 3^r$ is a power of three. Then we obtain recursive formulas for the power moments of Kloosterman sums with square arguments and for the even power moments of those in terms of the frequencies of weights in the codes. This is done via Pless power moment identity and by using the explicit expressions of Gauss sums for the symplectic groups $Sp(2n, q)$. The reader refers to [38] for the details on this and [12] for a general reference about linear codes.

One refers to [26]- [31], [36], [39] and [42] for many other papers of similar nature, which are related to other classical groups. Furthermore, many infinite families of binary and ternary linear codes were constructed in connection with double cosets with respect to certain maximal parabolic subgroups of classical groups. Then many infinite families of recursive formulas for (complete or incomplete) power moments of Kloosterman sums were obtained in terms of the frequencies of weights in those codes. These were also done via Pless power moment identity and by using the explicit expressions of exponential sums over those double cosets related to the evaluations of Gauss sums for the classical groups. For details on these, one refers to [32]- [35], [37] and [38].

With $q = 3^r$, for ease of notations, we let

$$G_1(q) = Sp(2, q),\ G_2(q) = Sp(4, q),$$

and let

$$N_1 = |G_1(q)| = q(q^2 - 1), \quad N_2 = |G_2(q)| = q^4(q^2 - 1)(q^4 - 1).$$

Here we construct two ternary linear codes $C(G_1(q))$ of length N_1 and $C(G_2(q))$ of length N_2, respectively associated with the symplectic groups $G_1(q)$ and $G_2(q)$.

By abuse of notations, for $i = 1, 2$, let $g_1, g_2, \ldots, g_{N_i}$ be a fixed ordering of the elements in the group $G_i(q)$. Also, for $i = 1, 2$, we put

$$v_i = (Trg_1, Trg_2, \ldots, Trg_{N_i}) \in \mathbb{F}_q^{N_i}.$$

Then, for $i = 1, 2$, the ternary linear code $C(G_i(q))$ is defined as

$$C(G_i(q)) = \{u \in \mathbb{F}_3^{N_i} \mid u \cdot v_i = 0\},$$

where the dot denotes the usual inner product in $\mathbb{F}_q^{N_i}$.

Delsarte's theorem says that $(B|_{\mathbb{F}_3})^\perp = tr(B^\perp)$, for any linear code B over $\mathbb{F}_q$, and from this it is immediate to see that

$$C(G_i(q))^\perp = \{c_i(a) = (tr(aTrg_1), \ldots, tr(aTrg_{N_i})) \mid a \in \mathbb{F}_q\}.$$

Then we can express the Hamming weights of $c_1(a)$ and $c_2(a)$ in terms of Kloosterman sums as in the following (*cf.* [38, Lemma 11, p. 87]):

$$w(c_1(a)) = \frac{2}{3}q(q^2 - 1 - K(\lambda; a^2)), \tag{13}$$

$$w(c_2(a)) = \frac{2}{3}q^4\{(q^2 - 1)(q^4 - 1) - (K(\lambda; a^2)^2 + q^3 - q)\}. \tag{14}$$

To get the results on the power moments of Kloosterman sums, we need the following Pless power moment identity.

Lemma 5 (*cf.* [12, Theorem 7.2.3, p. 257])**.** *Let B be an q-ary $[n, k]$ code, and let B_i(resp. $B_i^\perp$) denote the number of codewords of weight i in B(resp. in $B^\perp$). Then, for $h = 0, 1, 2, \ldots,$*

$$\sum_{j=0}^{n} j^h B_j = \sum_{j=0}^{min\{n,h\}} (-1)^j B_j^\perp \sum_{t=j}^{h} t! S(h, t) q^{k-t}(q - 1)^{t-j} \binom{n-j}{n-t}, \tag{15}$$

where $S(h, t)$ is the Stirling numbers of the second kind given by

$$S(h, t) = \frac{1}{t!} \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} j^h.$$

We now apply the Pless power moment identity in (15) to each $C(G_i(q))^{\perp}$, for $i = 1, 2$, in order to obtain the results about recursive formulas in Theorem 6 (*cf.* (19), (21)).
Then the left hand side of the identity in (15) is equal to

$$\sum_{a \in \mathbb{F}_q^*} w(c_i(a))^h, \tag{16}$$

with the $w(c_i(a))$ in each case given by (13) and (14).

For $i = 1$, (16) is

$$\begin{aligned}
&(\frac{2q}{3})^h \sum_{a \in \mathbb{F}_q^*} (q^2 - 1 - K(\lambda; a^2))^h \\
&= (\frac{2q}{3})^h \sum_{a \in \mathbb{F}_q^*} \sum_{j=0}^{h} (-1)^j \binom{h}{j} (q^2-1)^{h-j} K(\lambda; a^2)^j \qquad (17)\\
&= 2(\frac{2q}{3})^h \sum_{j=0}^{h} (-1)^j \binom{h}{j} (q^2-1)^{h-j} SK^j,
\end{aligned}$$

from which (19) follows.
Similarly, for $i = 2$, (16) equals

$$2(\frac{2q^4}{3})^h \sum_{j=0}^{h} (-1)^j \binom{h}{j} (q^6 - q^4 - q^3 - q^2 + q + 1)^{h-j} SK^{2j}, \tag{18}$$

from which (21) follows.
Here one has to separate the term corresponding to $j = h$ in (17) and (18), and note $dim_{\mathbb{F}_3} C(G_i(q))^{\perp} = r$.

Theorem 6. *Let $q = 3^r$. Then we have the following.*
(a) *For $h = 1, 2, \ldots,$*

$$\begin{aligned}
SK^h = &\sum_{j=0}^{h-1} (-1)^{h+j+1} \binom{h}{j} (q^2-1)^{h-j} SK^j \\
&+ q^{1-h} \sum_{j=0}^{min\{N_1,h\}} (-1)^{h+j} C_{1,j} \sum_{t=j}^{h} t! S(h,t) 3^{h-t} 2^{t-h-j-1} \binom{N_1 - j}{N_1 - t},
\end{aligned} \tag{19}$$

where $N_1 = |Sp(2,q)| = q(q^2-1)$, and $\{C_{1,j}\}_{j=0}^{N_1}$ is the weight distribution of the ternary linear code $C(Sp(2,q))$ given by

$$\begin{aligned}
C_{1,j} = &\sum \binom{q^2}{\nu_1, \mu_1} \binom{q^2}{\nu_{-1}, \mu_{-1}} \prod_{\beta^2 - 1 \neq 0 \ square} \binom{q^2+q}{\nu_\beta, \mu_\beta} \\
&\times \prod_{\beta^2 - 1 \ nonsquare} \binom{q^2 - q}{\nu_\beta, \mu_\beta} \ (j = 0, \cdots, N_1).
\end{aligned} \tag{20}$$

Here the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ and $\{\mu_\beta\}_{\beta \in \mathbb{F}_q}$ satisfying $\sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j$ and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta$. In addition,

$S(h,t)$ is the Stirling number of the second kind defined by

$$S(h,t) = \frac{1}{t!}\sum_{j=0}^{t}(-1)^{t-j}\binom{t}{j}j^h.$$

(b) *For $h = 1, 2, \ldots,$*

$$SK^{2h} = \sum_{j=0}^{h-1}(-1)^{h+j+1}\binom{h}{j}(q^6 - q^4 - q^3 - q^2 + q + 1)^{h-j}SK^{2j}$$
$$+ q^{1-4h}\sum_{j=0}^{min\{N_2,h\}}(-1)^{h+j}C_{2,j}\sum_{t=j}^{h}t!S(h,t)3^{h-t}2^{t-h-j-1}\binom{N_2-j}{N_2-t}, \quad (21)$$

where $N_2 = |Sp(4,q)| = q^4(q^2-1)(q^4-1)$, and $\{C_{2,j}\}_{j=0}^{N_2}$ is the weight distribution of the ternary linear code $C(Sp(4,q))$ given by

$$C_{2,j} = \sum\binom{q^4(\delta(2,q;0) + q^5 - q^2 - 3q + 3)}{\nu_0, \mu_0}$$
$$\times \prod_{\beta\in\mathbb{F}_q^\times}\binom{q^4(\delta(2,q;\beta) + q^5 - q^3 - q^2 - 2q + 3)}{\nu_\beta, \mu_\beta}(j = 0, \ldots, N_2). \quad (22)$$

Here the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ and $\{\mu_\beta\}_{\beta\in\mathbb{F}_q}$ satisfying $\sum_{\beta\in\mathbb{F}_q}\nu_\beta + \sum_{\beta\in\mathbb{F}_q}\mu_\beta = j$ and $s\sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = \sum_{\beta\in\mathbb{F}_q}\mu_\beta\beta$, and, for every $\beta \in \mathbb{F}_q$, $\delta(2,q;\beta) =| \{(\alpha_1,\alpha_2) \in (\mathbb{F}_q^\times)^2 \mid \alpha_1 + \alpha_1^{-1} + \alpha_2 + \alpha_2^{-1} = \beta\} |$.

Conclusion

In this paper, we gave a survey on Gauss sums for classical groups over finite fields. In addition, we presented three applications of them, namely to Hodges' Kloosterman sums, counting the number of elements in classical groups with given traces, and constructions of linear codes and power moments of Kloosterman sums.

Explicit expressions of the Gauss sums for classical groups were obtained by using three different ways:

• The first method was to use Bruhat decompositions with respect to certain maximal parabolic subgroups of the classical groups. We gave a sketch of proof for the case of the symplectic group $Sp(2n,q)$.

• The second one was to express the Gauss sum for G as the alternating sum of traces of Frobenius maps acting on Weyl group invariant part of certain cohomology groups of $(G/T)T$, where T is a maximal torus of G.

• The third one was to use some consequences of Deligne-Lusztig character theory of G and structure theory on maximal tori of G.

As the Gauss sums for classical groups were computed by using three completely different methods, we can naturally expect many interesting identities among them. The interested reader can compare, for example, the result for the Gauss sum

$$\sum_{g\in Sp(2n,q)}\psi(Trg)$$

in (8) with the ones in Theorem 1 of [2] and in Theorem 3.1 of [4]. Some of these were already noted in [3] and [4].

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References

[1] L. Carlitz, *Weighted quadratic partitions over a finite field*, Cand. J. Math. **5**, 317–323, 1953.

[2] H.-J. Chae and D. S. Kim, *L functions of some exponential sums of finite classical groups*, Math. Ann. **326**, 479–487, 2003.

[3] H.-J. Chae and D. S. Kim, *Identities arising from Gauss sums for finite classical groups*, J. Number Theory **128**, 2010–2024, 2008.

[4] H.-J. Chae and D. S. Kim, *Identities arising from Gauss sums for symplectic and orthogonal groups*, J. Korean Math. Soc. **47 (2)**, 409–424, 2010.

[5] P. Deligne, *Applications de la formule des traces aux sommes trigonométrique*, In: Cohomologie Etale, Séminaire de Géométrie Algébrique 4 1/2 by P. Deligne, Lecture Notes in Math. **569**, 168–232, 1977.

[6] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. Math. **103**, 103–161, 1976.

[7] G. Van der Geer, R. Schoof and M. Van der Vlugt, *Weight formulas for ternary Melas codes*, Math. Comp. **58**, 781–792, 1992.

[8] J. H. Hodges, *Weighted partitions for general matrices over a finite field*, Duke Math. J. **23**, 545–552, 1956.

[9] J. H. Hodges, *Weighted partitions for symmetric matrices in a finite field*, Math. Z. **66**, 13–24, 1956.

[10] J. H. Hodges, *Weighted partitions for skew matrices over a finite field*, Arch. Math. **8**, 16–22, 1957.

[11] J. H. Hodges, *Weighted partitions for Hermitian matrices over a finite field*, Math.Nachr. **17**, 93–100, 1958.

[12] W. C. Huffman and V. Pless, *Fundamentals of error-correcting codes*, Cambridge University Press, Cambridge, 2003.

[13] D. S. Kim, *Gauss sums for classical groups over a finite field*, In: Number Theory, Geometry and Related Topics, Iksan City, 1995, Pyungsan Inst. Math. Sci., Seoul, 97–101, 1996.

[14] D. S. Kim, *Gauss sums for symplectic groups over a finite field*, Monatsh. Math. **126**, 55–71, 1998.

[15] D. S. Kim, *Gauss sums for general and special linear groups over a finite field*, Arch. Math. (Basel) **69**, 297–304, 1997.

[16] D. S. Kim, *Gauss sums for* $O(2n+1, q)$, Finite Fields Appl. **4**, 62–86, 1998.

[17] D. S. Kim, *Gauss sums for* $O^-(2n, q)$, Acta Arith. **80**, 343 –365, 1997.

[18] D. S. Kim, *Gauss sums for $U(2n, q^2)$*, Glasgow Math. J. **40**, 79–95, 1998.

[19] D. S. Kim, *Gauss sums for $U(2n+1, q^2)$*, J. Korean Math. Soc. **34**, 871–894, 1997.

[20] D. S. Kim, *Exponential sums for symplectic groups and their applications*, Acta Arith. **88**, 155–171, 1999.

[21] D. S. Kim, *Exponential sums for $O^+(2n, q)$ and their applications*, Acta Math. Hungar. **91**, 79–97, 2001.

[22] D. S. Kim, *Exponential sums for $O^-(2n, q)$ and their applications*, Acta Arith. **97**, 67–86, 2001.

[23] D. S. Kim, *Exponential sums for $O(2n+1, q)$ and their applications*, Glasg. Math. J. **43**, 219–235, 2001.

[24] D. S. Kim, *Sums for $U(2n, q^2)$ and their applications*, Acta Arith. **101**, 339–363, 2002.

[25] D. S. Kim, *Hodges' Kloosterman sums*, In: Number Theory and Related Topics, Seoul, 1998, Yonsei Univ. Inst. Math. Sci., Seoul, 119–133, 2000.

[26] D. S. Kim, *Codes associated with special linear groups and power moments of multi-dimensional Kloosterman sums*, Ann. Mat. Pura Appl. **190 (4)**, 61–76, 2011.

[27] D. S. Kim, *Codes associated with $O^+(2n, 2^r)$ and power moments of Kloosterman sums*, Integers **12 (2)**, 237–257, 2012.

[28] D. S. Kim, *Codes associated with orthogonal groups and power moments of Kloosterman sums*, J. Comb. Number Theory **3**, 65–81, 2011.

[29] D. S. Kim, *Codes associated with $O(3, 2^r)$ and power moments of Kloosterman sums with trace one arguments*, Proc. Jangjeon Math. Soc. **15 (2)**, 171–181, 2012.

[30] D. S. Kim, *Ternary codes associated with $O(3, 3^r)$ and power moments of Kloosterman sums with trace nonzero square arguments*, Math. Commun. **17**, 241–255, 2012.

[31] D. S. Kim, *Ternary codes associated with $O^-(2n, q)$ and power moments of Kloosterman sums with square arguments*, Lith. Math. J. **51 (4)**, 507–521, 2011.

[32] D. S. Kim, *An infinite family of recursive formulas generating power moments of Kloosterman sums with trace one arguments: $O(2n+1, 2^r)$ Case*, Integers **13**, #A32, 2013.

[33] D. S. Kim, *Infinite families of recursive formulas generating power moments of ternary Kloosterman sums with square arguments arising from symplectic groups*, Adv. Math. Commun. **3**, 167–178, 2009.

[34] D. S. Kim, *Infinite families of recursive formulas generating power moments of Kloosterman sums: $O^-(2n, 2^r)$ Case*, Math. Slovaca **63 (4)**, 733–758, 2013.

[35] D. S. Kim, *Infinite families of recursive formulas generating power moments of ternary Kloosterman sums with square arguments associated with $O^-(2n, q)$*, J. Korean Math. Soc. **48 (2)**, 267–288, 2011.

[36] D. S. Kim, *Recursive formulas generating power moments of multi-dimensional Kloosterman sums and m-multiple power moments of Kloosterman*, Algebra Discrete Math. **19 (2)**, 213–228, 2015.

[37] D. S. Kim, *Construction of recursive formulas generating power moments of Kloosterman sums: $O^+(2n, 2^r)$ Case*, J. Korean Math. Soc. **57 (3)**, 585–602, 2020.

[38] D. S. Kim, *Recursive formulas generating power moments of ternary Kloosterman sums with trace nonzero square arguments: $O(2n+1, 3^r)$ Case*, Montes Taurus J. Pure Appl. Math. **1 (2)**, 21–41, 2019; Article ID: MTJPAM-D-19-00002.

[39] D. S. Kim and J. H. Kim, *Ternary codes associated with symplectic groups and power moments of Kloosterman sums with square arguments*, Proc. Jangjeon Math. Soc. **15 (1)**, 79–90, 2012.

[40] D. S. Kim and I.-S. Lee, *Gauss sums for $O^+(2n, q)$*, Acta Arith. **78**, 75–89, 1996.

[41] D. S. Kim and Y.-H. Park, *Gauss sums for orthogonal groups over a finite field of characteristic two*, Acta Arith. **82**, 331–357, 1997.

[42] D. S. Kim and S.-H. Yang, *A recursive formula for power moments of 2-dimensional Kloosterman sums associated with general linear groups*, Italian J. Pure Appl. Math. **34**, 7–16, 2015.

[43] H.D. Kloosterman, *On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$*, Acta Math. **49**, 407–464, 1926.

[44] Y. Li and S. Hu, *Gauss sums over some matrix groups*, J. Number Theory **13**, 2967–2976, 2012.

[45] R. Lidl and H. Niederreiter, *Finite fields*, Encyclopedia of Math. Appl. 20, Cambridge University Press, Cambridge, 1987.

[46] M. Moisio, *On the moments of Kloosterman sums and fibre products of Kloosterman curves*, Finite Fields Appl. **14**, 515–531, 2008.

[47] C. Perret-Gentil-dit-Maillard, *Probabilistic aspects of short sums of trace functions over finite fields*, Doc. Sci. Thesis, ETH Zürich, 2016.

[48] H. Saito, *A generalization of Gauss sums and its applications to Siegel modular forms and L functions associated with the vector space of quadratic forms*, J. Reine Angew. Math. **416**, 91–142, 1991.

[49] H. Salié, *Über die Kloostermanshen summen $S(u; v; q)$*, Math. Z. **34**, 91–109, 1932.

[50] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36**, 173–207, 1976.

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A study on Frobenius Euler-type Simsek numbers and polynomials

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In this work, we give some results and local approximation properties such as Lipshtsz class for a generalization Szasz-type operators including the generating function of Frobenius Euler-type Simsek polynomials.

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KEYWORDS: Apostol-type polynomials, Generating function, Approximation by polynomials, Approximation by positive operators

Introduction

The Frobenious Euler-type Simsek polynomials $\ell_n(x;v)$ given by the following generating function:

$$F_\ell\left(x;w,v\right) := \frac{w^v}{\prod\limits_{j=0}^{v-1}\left(e^w - j\right)} e^{wx} = \sum_{n=0}^{\infty} \ell_n(x;v)\frac{w^n}{n!} \tag{1}$$

which were recently introduced and investigated by Simsek in [6]. Substituting $x = 0$ into (1) gives the generating function of the Apostol-type numbers $\ell_n(v)$ as follows:

$$F_\ell\left(w,v\right) := \frac{w^v}{\prod\limits_{j=0}^{v-1}\left(e^w - j\right)} = \sum_{n=0}^{\infty} \ell_n(v)\frac{w^n}{n!} \tag{2}$$

which is equivalent to the special case $f(w) = e^w$, $\overrightarrow{x_v} = (0, 1, 2, \ldots, v-1)$ and $\overrightarrow{y_v} = (1, 1, \ldots, 1)$ of the following meromorphic function:

$$F_1\left(w;\overrightarrow{x_v},\overrightarrow{y_v}\right) = \frac{w^v}{h\left(w;\overrightarrow{x_v},\overrightarrow{y_v}\right)},$$

where

$$h\left(w;\overrightarrow{x_v},\overrightarrow{y_v}\right) = \prod_{j=0}^{v-1}\left(f(w) - x_j\right)^{y_j},$$

$f(w)$ is an analytic function such that $w \in \mathbb{R}$ (or $\mathbb{C}$); $\overrightarrow{x_v} = (x_0, x_1, \ldots, x_{v-1})$ and $\overrightarrow{y_v} = (y_0, y_1, \ldots, y_{v-1})$ are v-tuples such that $v \in \mathbb{N}$ and $x_j, y_j \in \mathbb{R}$ with $j = 0, 1, \ldots, v-1$. See, for detail, [6].

The $\ell_n(x;2)$ polynomials are defined at the following equation:

$$\frac{w^2}{e^w - 1} e^{w(x-1)} = \sum_{n=0}^{\infty} \ell_n(x;2)\frac{w^n}{n!}. \tag{3}$$

The Taylor expansion of the generating function of $\ell_n(x;2)$ is defined by the following expression:

$$\frac{w}{e^w-1}+\frac{w^2(x-1)}{e^w-1}+\frac{w^3(x-1)^2}{2(e^w-1)}+\frac{w^4(x-1)^3}{6(e^w-1)}+\frac{w^5(x-1)^4}{24(e^w-1)}+\frac{w^6(x-1)^5}{120(e^w-1)}$$
$$+O\left((x-1)^6\right).$$

By using the Taylor expansion of $\ell_n(x;2)$, this series converges everywhere and positive at $x \in (1,\infty)$.

The Szasz type operators which involving the generating function of $\ell_n(x;2)$ were defined to be as:

$$\mathfrak{L}_n(f;x)=(e^2-e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(x;2)}{k!}f\left(\frac{k}{n}\right). \tag{4}$$

The moment and second-order central moment functions for the operator in equation (4) were obtained as follows:

$$\mathfrak{L}_n(1;x)=1,$$
$$\mathfrak{L}_n(s;x)=x-\frac{e}{(e-1)n},$$
$$\mathfrak{L}_n(s^2;x)=x^2+\frac{3e-1}{e-1}\frac{x}{n}+\frac{-e^2+4e-1}{n^2(e-1)^2},$$

and

$$\mathfrak{L}_n((s-x)^2;x)=\frac{x}{n}+\frac{-e^2+4e-1}{n^2(e-1)^2}.$$

Remark 1. *The convergence of the operator using moment and second-order moment functions and the approximation error estimation with the help of the modulus of continuity were investigated in* [1].

Main results

In this section, we estimate the approximation error of the operator defined in (7) using approximation tools such as the Lipsithz class and the Peetre-K functional.

The definition of the Lipschitz class is defined to be

Definition 1. *Lipschitz class of order α, express $Lip_1(\alpha;K)$ $(0<\alpha\leq 1,\ K>0)$, is defined by*

$$Lip_1(\alpha;K):=\left\{f\in C([0,1]):|f(t)-f(x)|\leq K|t-x|^{\alpha},\ t,x\in[0,1]\right\},$$

where $C([0,1])$ is the set of spaces of continuous functions.

By using the monotonic property of $L_n(f,x)$, we obtain

$$|\mathfrak{L}_n(f;x)-f(x)|\leq K\mathfrak{L}_n(|t-x|^{\alpha};x).$$

By applying Hölder inequality at the above inequality, we give

$$|\mathfrak{L}_n(f;x)-f(x)|\leq K\left(\mathfrak{L}_n\left((e_1-x)^2;x\right)\right)^{\alpha/2}.$$

The last inequality above gives the following theorem:

Theorem 2. *Let $f \in Lip_1(\alpha; K)$. For $x \in [0,1]$, we have*

$$|L_n(f;x) - f(x)| \leq K\sqrt{\varsigma_n^\alpha(x)},$$

where $\varsigma_n(x) = \mathfrak{L}_n((s-x)^2; x)$.

The aforementioned lemmas provide some characteristics of the central moment functions of $\mathfrak{L}_n(f;x)$:

Corollary 3. *With respect to the operators $\mathfrak{L}_n(f;x)$, we give*

$$\mathfrak{L}_n((e_1 - e_0 x); x) \leq \frac{1.58}{n}$$

$$\mathfrak{L}_n((e_1 - e_0 x)^2; x) \leq \frac{1}{n} + \frac{0.841}{n^2}.$$

Corollary 4. *The following expressions hold true:*

$$\lim_{n\to\infty} n\mathfrak{L}_n((e_1 - e_0 x); x) = 1,$$

$$\lim_{n\to\infty} n\mathfrak{L}_n((e_1 - e_0 x)^2; x) = x.$$

Conclusion

In this paper, a theorem for the local approximation property for Szasz-type operators involving generating functions of Frobenious-Euler type Simsek polynomials given in [1], and two new results for the operator with the help of central moment functions are obtained.

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References

[1] E. Agyuz, *Identities derived from a particular class of generating functions for Frobenius-Euler type Simsek numbers and polynomials*, Filomat **38 (5)**, 1531–1545, 2024.

[2] Y. Simsek, *Generating functions for q-Apostol type Frobenius-Euler numbers and polynomials*, Axioms **1**, 395–403, 2012.

[3] D. S. Kim and T. Kim, *Some new identities of Frobenius-Euler numbers and polynomials*, J. Ineq. Appl. **2012**, 2012, Article ID: 307.

[4] Y. Simsek, *Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their applications*, Fixed Point Theory Appl. **87**, 343–355, 2013.

[5] T. Kim, D. S. Kim, D. V. Dolgy and J.-J. Seo, *Bernoulli polynomials of the second kind and their identities arising from umbral calculus*, J. Nonlinear Sci. Appl. **9**, 860–869, 2016.

[6] Y. Simsek, *Applications of constructed new families of generating-type functions interpolating new and known classes of polynomials and numbers*, Math. Methods Appl. Sci. **44 (14)**, 11245–11268, 2021.

[7] Y. Simsek, *Applications of constructed new families of generating-type functions interpolating new and known classes of polynomials and numbers*, Math. Methods Appl. Sci. **44 (14)**, 11245–11268, 2021.

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Observations on the Dedekind zeta function at negative integers with their applications

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The motivation of this presentation is to survey and study the connections between the negative integer values of the known Dedekind zeta function related to Bernoulli numbers on different finite fields.

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KEYWORDS: Riemann zeta function, Dirichlet L-functions, Dedekind zeta, Bernoulli numbers, Norm of ideals, Rings, Character

Introduction

The norm of an ideal is a generalization of a norm of an element in the field extension. Let $\mathbb{Z}$ be the ring of integers. The norm of a nonzero ideal $\mathfrak{a}$ of a number ring R is simply the size of the finite quotient ring $R/\mathfrak{a}$.

The Riemann zeta function can also be interpreted as the sum of $N\mathfrak{a}^{-s}$ over all ideals, $\mathfrak{a}=(n)$ of the ring $\mathbb{Z}$. Thus the norm of $\mathfrak{a}=(n)$ is equal to $N(n)=|n|$, which is the number of elements $\mathbb{Z}/n\mathbb{Z}$. Consequently, summation over ideals means that n and $-n$ (for $n\in\mathbb{N}$) only contribute n^{-s}. The number $N\mathfrak{a}$ is a norm of an ideal $\mathfrak{a}$.

For instance, let $\mathbb{K}=\mathbb{Q}(i)$ and $\mathbb{Z}[i]$ be a ring of Gaussian integers. Thus, each ideal $(x+iy)$ $(x,y\in\mathbb{Z})$ has unique representative in the first quadrant. Hence one has

$$\zeta_{\mathbb{K}}(s)=\sum_{\mathfrak{a}\neq 0}N\mathfrak{a}^{-s}=\sum_{\substack{x,y\geq 0\\(x,y)\neq 0}}\frac{1}{(x^2+y^2)^s}. \tag{1}$$

Similar to the Riemann zeta function, unique factorization in $\mathbb{Z}[i]$ implies that the function on $\mathbb{Z}[i]$ admits the following Euler factorization:

$$\zeta_{\mathbb{K}}(s)=\prod\frac{1}{1-N\pi^{-s}}. \tag{2}$$

There are exactly two primes π above primes. Due to these primes, in $\mathbb{Z}[i]$, we have the Euler product not only for $\mathfrak{p}\equiv 1 \pmod 4$:

$$\prod_{\mathfrak{p}\equiv 1 \pmod 4}\frac{1}{(1-\mathfrak{p}^{-s})^2},$$

but also, for the primes $\mathfrak{p}\equiv 3 \pmod 4$, remain inert in $\mathbb{Z}[i]$ and have norm $\mathfrak{p}^2$,

$$\prod_{\mathfrak{p}\equiv 3 \pmod 4}\frac{1}{1-\mathfrak{p}^{-2s}}$$

(*cf.* [5, p. 13-14]).

And we know that,

$$\zeta_{\mathbb{K}}(s) = \prod_{\mathfrak{p}\equiv 3 \pmod 4} \frac{1}{1-2^{-s}} \prod_{\mathfrak{p}\equiv 1 \pmod 4} \frac{1}{(1-\mathfrak{p}^{-s})^2} \prod_{\mathfrak{p}\equiv 3 \pmod 4} \frac{1}{1-\mathfrak{p}^{-2s}}.$$

Thus

$$\zeta_{\mathbb{K}}(s) = \zeta(s) \prod_{\mathfrak{p}\equiv 1 \pmod 4} \frac{1}{(1-\mathfrak{p}^{-s})} \prod_{\mathfrak{p}\equiv 3 \pmod 4} \frac{1}{(1+\mathfrak{p}^{-s})}. \tag{3}$$

Therefore,

$$\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi), \tag{4}$$

where $L(s,\chi)$ denotes the Dirichlet L-Series with the character $(\frac{-4}{\circ})$, which is defined with the Euler product as follows:

$$L(s,\chi) = \prod_{\mathfrak{p}:prime} \frac{1}{1-\chi(\mathfrak{p})\mathfrak{p}^{-s}}, \tag{5}$$

where $s > 1$ (*cf.* [5, p. 13-14]). When $\chi(n)$ is a the Dirichlet character, which is a multiplicative function, one also has

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}. \tag{6}$$

Using the following well-known results

$$L(1,\chi) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$$

and

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{x^2+1},$$

it is easy to see that the function $L(s,\chi)$ is converges at $s = 1$. It also converges to a non-zero limit when $s \to 1$. Multiplying equation (4) by $s - 1$ and $s \to 1$, one easily has

$$\lim_{s\to 1}(s-1)\zeta_{\mathbb{K}}(s) = \frac{\pi}{4} \tag{7}$$

in $\mathbb{K} = \mathbb{Q}(i)$ (*cf.* [5, p. 13-14]).

We now give the Dedekind zeta function. This function was firstly defined by German mathematician Julius Wilhelm Richard Dedekind (6 October 1831-12 February 1916). This function is a member of the Dirichlet series. This is defined by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}\neq(0)} N\mathfrak{a}^{-s}, \tag{8}$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, $\mathbb{K}$ is an algebraic number field (number field). An algebraic number field is an extension field $\mathbb{K}$ of the field of rational numbers $\mathbb{Q}$ such that the field extension $\mathbb{K}/\mathbb{Q}$ has finite degree. $\mathbb{K}$ is a field that contains $\mathbb{Q}$. $\mathfrak{a}$ ranges through the non-zero ideals of the ring of integers $\mathcal{O} \subseteq \mathbb{Z}$, $\mathcal{O}_{\mathbb{K}}$ of $\mathbb{K}$ and $N\mathfrak{a}$ denotes the absolute norm of $\mathfrak{a}$ (which is equal to both the index $[\mathcal{O}_{\mathbb{K}} : \mathfrak{a}]$ of $\mathfrak{a}$ in $\mathcal{O}_{\mathbb{K}}$ or equivalently the cardinality of quotient ring $\mathcal{O}_{\mathbb{K}}/\mathfrak{a}$). This sum converges absolutely for all complex numbers s with real part $\mathrm{Re}\, s > 1$ (*cf.* [5, 9, 10]).

Substituting $\mathbb{K}=\mathbb{Q}$ into (8), the Dedekind zeta function $\zeta_{\mathbb{K}}(s)$ reduces to that of the Riemann zeta function; that is

$$\zeta(s) := \zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $\operatorname{Re} s > 1$.

This function has the following unique factorization into prime ideals:

$$\zeta_K(s) = \prod_{\mathfrak{p} \neq (0)} \frac{1}{1 - N\mathfrak{p}^{-s}},$$

where the product is over all prime ideals $\mathfrak{p} \neq (0)$.

In 1937, Siegel proved that $\zeta_K(2m)$ satisfies the following relation:

$$\zeta_K(2m) = rational \times \frac{\pi^{2dm}}{\sqrt{D_{\mathbb{K}}}},$$

where m is a positive integer, $\mathbb{K}$ is any totally real number field with discriminant $D_{\mathbb{K}}$ and d is the degree of $\mathbb{K}$ over $\mathbb{Q}$. When $\mathbb{K}$ is real quadratic number field, the values of the $\zeta_K(2m)$ are related to Bernoulli numbers similar to the

$$\zeta(2m) = (-1)^m \frac{(2\pi)^{2m} B_{2m}}{2\,(2m)!}$$

(*cf.* [3]), where B_m denotes the Bernoulli numbers, which are defined by means of the following generating function:

$$\sum_{m=0}^{\infty} B_m t^m = \frac{t}{e^t - 1}$$

with $|t| < 2\pi$.

Let $\mathbb{K} = \mathbb{Q}(\alpha)$ where α satisfies a polynomial of degree n with n real roots. D is a discriminant of the field $\mathbb{K}$. Substituting $s = 1 - 2m$ and $s = -2m$ with $m \in \mathbb{N}$ into (4), and combining the following well-known results for the Riemann zeta function at the negative integers

$$\zeta(-2m) = -\frac{B_{2m+1}}{2m+1} = 0$$

since $B_{2m+1} = 0$ for $m \in \mathbb{N}$ and

$$\zeta(1-2m) = -\frac{B_{2m+2}}{2m+2},$$

where $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0,...,$ we have

$$\zeta_{\mathbb{K}}(1-2m) = \frac{B_{2m}}{4m^2} D^{2m-1} \sum_{n=1}^{D} \chi(n) B_{2m}\left(\frac{i}{D}\right), \tag{9}$$

and

$$\zeta_{\mathbb{K}}(1-2m) = \frac{(-1)^m (2m-1)!}{2^{2m-1}\pi^{2m}} D^{2m-1/2} \zeta_{\mathbb{K}}(2m) \tag{10}$$

where $i^2 = -1$ see, for detail [8].

We modified equation (9) as follows:

$$\zeta_{\mathbb{K}}(1-2m) = \frac{B_{2m}}{4m^2} D^{2m-1} \sum_{n=1}^{D} \chi(n) \sum_{v=0}^{2m} \binom{2m}{v} \frac{i^v}{D^v} B_v.$$

Some special known values of the $\zeta_{\mathbb{K}}(1-2m)$ are given as follows:
Putting $m=1$ in (9) and (10), and using

$$\zeta(1-2m) = \frac{(-1)^m.(2m-1)!}{2^{2m-1}\pi^{2m}} \zeta(2m), \tag{11}$$

and $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ with $p \equiv 1 \pmod 4$ a prime number, we also have the following well-known formula:

$$\zeta_{\mathbb{K}}(-1) = \frac{1}{24p} \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) j^2,$$

where $\left(\frac{j}{p}\right)$ is the Legendre-Jacobi symbol. Since

$$B_2(x) = x^2 - x + \frac{1}{6},$$

we have

$$\zeta_K(-1) = \frac{1}{24D} \sum_{n=1}^{D-1} \chi(n) n^2$$

(*cf.* [8]). Thus, for $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ and $\mathbb{K} = \mathbb{Q}(\sqrt{13})$, we have the following values of $\zeta_K(-1)$, respectively:

$$\begin{aligned} \zeta_K(-1) &= \frac{1^2 - 2^2 - 3^2 + 4^2}{120} \\ &= \frac{1}{30}. \end{aligned}$$

and

$$\begin{aligned} \zeta_K(-1) &= \frac{1^2 - 2^2 + 3^2 + 4^2 + 5^2 - 6^2 - 7^2 - 8^2 + 9^2 + 10^2 - 11^2 + 12^2}{13.24} \\ &= \frac{1}{6}. \end{aligned}$$

Is it possible to give general formula for $\zeta_K(-1)$ on $\mathbb{K} = \mathbb{Q}(\sqrt{p})$, p is prime number?

Conclusion

The main goal of this study is to investigate and examine the relations between well-known some of values of on the Dedekind zeta function at the negative integers on different finte fields and related ideals of the ring of integers. Our future project will be combined these relations and other certain family of special numbers and polynomials with the aid of characters and norms of the ideals.

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References

[1] M. Alkan and Y. Simsek, *The families of L-Series associated with decomposition of the generating functions*, Filomat **30 (7)**, 1789–1799, 2016.

[2] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli numbers and zeta functions*, Springer, 2014.

[3] S. Banerjee, K. Chakraborty and A. Hoque, *An analogue of Wilton's formula and values of Dedekind zeta functions*, https://arxiv.org/pdf/1611.08693.pdf

[4] D. Kramer, *On the values at ıntegers of the dedekind zeta function of a real quadratic field source*, Trans. Am. Math. Soc. 1987.

[5] F. Lemmermeyer, *Class field theory*, Bilkent University, Dept. Mathematics, Ankara, Turkey, 2007.

[6] K. Martin, *Onunıque factorization and principalization in number fields*, Proc. Am. Math. Soc. 2011.

[7] Y. Simsek and H. Ozden, *Unified representation of the family of L-functions*, J. Inequal. Appl. **2013**, 2013.

[8] D. Zagier, *On the values at negative integers of the zeta-function of a real quadratic field*, L'Enseignement Mathematique **22**, 1976.

[9] https://en.wikipedia.org/wiki/Algebraic_number_field, (Accession date: 09 Jan 2024).

[10] https://en.wikipedia.org/wiki/Dedekind_zeta_function, (Accession date: 09 Jan 2024).

[11] https://en.wikipedia.org/wiki/Ideal_norm#Absolute_norm, (Accession date: 09 Jan 2024).

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Applying the operator $Y_{\lambda,\beta}(f; a, b)$ to Derangement polynomials

Elif Bozo [*1] *and Yilmaz Simsek* [2]

The aim of this presantation is to investigate applications of the operator $Y_{\lambda,\beta}(f; a, b)$ to the certain family of special polynomials. By using these applications, we give some new formulas including the Derangement polynomials.

2020 MSC: 05A15, 11B75, 11B83, 60G50, 11B37

KEYWORDS: Generating functions, Derangement numbers, Finite sums, Abel polynomials, Harmonic numbers, Operator

Introduction

Operator

Let a and b be real parameters. Let λ and β be real or complex parameters. In this section, we give applications of the operator $Y_{\lambda,\beta}(f : a, b)$, which introduced by the second author (*cf.* [6]):

$$Y_{\lambda,\beta}[f : a, b](x) = \lambda E^a[f](x) + \beta E^b[f](x), \tag{1}$$

where $E^a[f]$ is well-known shift:

$$E^a[f](x)] = f(x + a).$$

In the special case $\lambda = 1, \beta = 0$ and $b = 0$,we have

$$Y_{1,0}[f : a, 0](x) = E^a[f](x). \tag{2}$$

Let $k \in \mathbb{N}$. Since

$$Y^k_{\lambda,\beta}[f; a, b] = Y_{\lambda,\beta}[f; a, b]\left(Y^{k-1}_{\lambda,\beta}[f; a, b]\right), \tag{3}$$

we have

$$Y^k_{\lambda,\beta}[f; a, b] = \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j}\beta^j f(x + jb + (k - j)a), \tag{4}$$

(*cf.* [6]).

A Derangement of an ordered set of objects is a way of rearranging the objects so that none appears in its original position. A Derangement of $(1, 2, 3, \ldots, n)$, for example, is an arrangement $(x_1, x_2, x_3, \ldots, x_n)$ of the first n integers in which $x_i = i$ never occurs.(*cf.* [8, p. 96]).

The derangement numbers d_n, which have many applications in enumerative combinatorics analysis, are given by meas of the following generating function:

$$T(t) = \frac{e^{-t}}{1 - t} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}. \tag{5}$$

(*cf.* [2, p. 171], [7], [8, p. 97]).

By using (5), for $n \in \mathbb{N}_0$, we have

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}, \tag{6}$$

(*cf.* [2, p. 171], [7], [8, p. 97]).

The derangement polynomials are defined by means of the generating function:

$$\frac{e^{(x-1)t}}{1-t} = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!},$$

which yields

$$\begin{aligned} D(x) &= \sum_{k=0}^{n} \binom{n}{k} d_k x^{n-k} \\ &= n! \sum_{k=0}^{n} \frac{(x-1)^k}{k!} \end{aligned} \tag{7}$$

(*cf.* [2, p. 171], [7], [8, p. 97]).

Main results

In this section, by applying the operator $Y_{\lambda,\beta}(f : a, b)$ to the derangement polynomials $D(x)$ which are given in (7), we derive some formulas.

By applying (*cf.* [6]) to (7), we get

$$Y_{\lambda,\beta}[D_n : a, b](x) = \lambda E^a (D_n(x)) + \beta E^b (D_n(x))$$

Thus, we have

$$Y_{\lambda,\beta}[D_n : a, b](x) = \lambda D_n(x + a) + \beta D_n(x + b).$$

By applying the above operator application k times consecutively, the following result is obtained with the help of equation (4):

$$Y^k_{\lambda,\beta} [D_n : a, b] (x) = \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \beta^j D_n(x + jb + (k - j)a).$$

Thus we have

$$Y^k_{\lambda,\beta} [D_n : a, b] (x) = n! \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \beta^j \sum_{v=0}^{n} \frac{(x + jb + (k - j)a - 1)^v}{v!}.$$

After some elemantary calculations, we arrive at the following theorem:

Theorem 1. *Let $T \subseteq \mathbb{N}$. Then we have*

$$\begin{aligned} & n! \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \beta^j \sum_{v=0}^{n} \frac{(x + jb + (k - j)a - 1)^v}{v!} \\ = & \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \beta^j D_n(x + jb + (k - j)a). \end{aligned} \tag{8}$$

Substituting $a = 0$ and $b = 1$ into (8), we get

$$\sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \beta^j D_n(x + j) = n! \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \beta^j \sum_{v=0}^{n} \frac{(x + j - 1)^v}{v!}.$$

Conclusion

The goal of this work is to give applications of the operator $Y_{\lambda,\beta}(f;a,b)$. There are many application s of this operator. By applying the this operator k times consecutively to the Derangement polynomials, with the aid of result, we derived some new formulasof the Derangement polynomials. Our future project will be deeaply study on this operator with its applications.

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References

[1] E. Bozo and Y. Simsek, *Some new formulas derived from generating function for derangement numbers*, In: Proceedings Book of the 13th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP 2023) Dedicated To Professor Yilmaz Simsek on the occasion of his 60th anniversary (Ed by M. Alkan, I.Kucukoglu and O. Öneş), Antalya, Turkey, March 11-13, 2023, pp. 157–160; ISBN: 978-625-00-1128-7

[2] C. A. Charalambides, *Enumerative combinatorics*, Chapman and Hall CRC London, New York, 2002.

[3] P. C. Consul and F. Famoye, *Lagrangian probability distributions*, Birkhauser, Boston, 2006.

[4] H. K. Kim, D. S. Kim and T. Kim, *Identities involving degenerate harmonic and degenerate hyperharmonic numbers*, 2023; ArXiv:2301.02861.

[5] S. H. Rim, T. Kim and S. S. Pyo, *Identities between harmonic, hyperharmonic and Daehee numbers*, J. Inequal. Appl. **2018**, 2018; Article ID: 168.

[6] Y. Simsek, *Some new families of special polynomials and numbers associated with finite operators*, Symmetry **12 (2)**, 2020; https://doi.org/10.3390/sym12020237

[7] R. P. Stanley and S. P. Fomin, *Enumerative combinatorics* (Volume 2, Edition 1), Cambridge University Press, 2001.

[8] W. D. Wallis and J. C. George, *Introduction to combinatorics*, CRC Press, 2011.

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Remarks on q-Hardy-Berndt type sums

Elif Cetin

In the study mentioned as [28], Simsek defined a new function called $\mathscr{F}(t,q)$. He used this function to create generating functions for the q-Hardy-Berndt type sums, which are related to classical Hardy-Berndt sums, Simsek sums, and various other well-known special sums. This study aims to establish connections between q-Hardy-Berndt type sums and some finite special sums, defined in [9], by employing the $\mathscr{F}(t,q)$ function and considering specific cases of the generating functions, as presented in [28].

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KEYWORDS: Special finite sums, Generating functions, Hardy sums, Simsek sums, $C(h,k;1)$ sums, $B_1(h,k)$ sums

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References

[1] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Springer-Verlag, New York, 1976.

[2] T. M. Apostol and T. H. Vu, *Elementary proofs of Berndt's reciprocity laws*, Pasific J. Math. **98**, 17–23, 1982.

[3] M. Beck, *Geometric proofs of polynomial reciprocity laws of Carlitz, Berndt, and Dieter*, In: Diophantine analysis and related fields, Sem. Math. Sci. **35**, 11–18, 2006.

[4] B. C. Berndt, *Analytic Eisenstein series, Theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303/304**, 332–150, 1978.

[5] B. C. Berndt and U. Dieter, *Sums involving the greatest integer function and Riemann Stieltwes integration*, J. Reine Angew. Math. **337**, 208–220, 1982.

[6] B. C. Berndt and R. J. Evans, *Problem E2758*, American Math. Monthly **86**, p. 128, 1979; **87**, 404, 1980.

[7] B. C. Berndt and L. A. Goldberg, *Analytic properties of arithmetic sums arising in the theory of the classical Theta-functions*, SIAM. J. Math. Anal. **15**, 143–150, 1984.

[8] L. Carlitz, *Some polynomials associated with Dedekind sums*, Acta Math. Sci. Hungar **26**, 311–319, 1975.

[9] E. Cetin, Y. Simsek and I. N. Cangul, *Some special finite sums related to the three-term polynomial relations and their applications*, Adv. Differ. Equ. **2014**, 2014; Article ID: 283.

[10] E. Cetin, *Analytic properties of the Sum* $B_1(h,k)$, Math. Comput. Appl. **21 (3)**, 2016, Article ID: 31.

[11] E. Cetin, *A note on Hardy type sums and Dedekind sums*, Filomat **30 (4)**, 977–983, 2016.

[12] E. Cetin, *Remarks on special sums associated with Hardy sums*, In: Proceedings Book of the Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2018) (Ed. by Y. Simsek), Antalya, Turkey, October 26-29, 2018, pp. 153-156; ISBN: 978-86-6016-036-4.

[13] E. Cetin, *A Note on trigonometric identities of the special finite sums*, In: Proceedings Book of the 13th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP 2023) (Ed. by M. Alkan, I. Kucukoglu and O. Ones), Antalya, Turkey, March 11–13, 2023, pp. 161-166; ISBN: 978-625-00-1128-7.

[14] E. Cetin, *Identities for a special finite sum related to the Dedekind sums and Fibonacci numbers*, GU J Sci, Part A **10 (2)**, 232–240, 2023.

[15] R. Dedekind, *Erläuterungen zu den Fragmenten XXVIII*, In: Collected works of Bernhard Riemann, Dover Publ., New York, pp. 466-478, 1953.

[16] L. A. Goldberg, *Transformation of Theta-functions and analogues of Dedekind sums*, Thesis, University of Illinois Urbana, 1981.

[17] H. Rademacher and E. Grosswald, *Dedekind sums*, Carus Mathematical Monographs, The Mathematical Association of America, 1972.

[18] G. H. Hardy, *On certain series of discontinues functions connected with the modular functions*, Quart. J. Math. **36**, 93–123, 1905.

[19] T. Koshy, *Fibonacci and Lucas identities*, In: Fibonacci and Lucas Numbers with Applications, John Wiley and Sons, New York, NY, USA, pp. 69-100, 2001.

[20] J. L. Meyer, *Symmetric arguments in the Dedekind sum*, Syracuse University, Syracuse, NY, USA, 2002.

[21] M. R. Pettet and R. Sitaramachandraro, *Three-term relations for Hardy sums*, J. Number Theory **25**, 328–339, 1989.

[22] H. Rademacher, *Generalization of the reciprocity formula for Dedekind sums*, Duke Math. J. **21**, 391–397, 1954.

[23] Y. Simsek, *A note on Dedekind sums*, Bull. Cal. Math. Soci. **85**, 567–572, 1993.

[24] Y. Simsek, *Theorems on three-term relations for Hardy sums*, Turkish J. Math. **22**, 153–162, 1998.

[25] Y. Simsek, *Relations between theta-functions Hardy sums Eisenstein series and Lambert series in the transformation formula of* $\log \eta_{g,h}(z)$, J. Number Theory **99**, 338–360, 2003.

[26] Y. Simsek, *On generalized Hardy sums* $s_5(h,k)$, Ukrainian Math. J. **56 (10)**, 1434–1440, 2004.

[27] Y. Simsek, *q-Dedekind type sums related to q-zeta function and basic L-series*, J. Math. Anal. Appl. **318**, 333–351, 2006.

[28] Y. Simsek, *q-Hardy–Berndt type sums associated with q-Genocchi type zeta and q-l-functions*, Nonlinear Anal. **71**, 377–395, 2009.

[29] Y. Simsek, *On Analytic properties and character analogs of Hardy sums*, Taiwanese J. Math. **13**, 253–268, 2009.

[30] Y. Simsek, *Special functions related to Dedekind-type DC-sums and their applications*, Russ. J. Math. Phys. **4**, 495–508, 2010.

[31] R. Sitaramachandrarao, *Dedekind and Hardy sums*, Acta Arith. XLVIII, 1978.

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Properties of multilinear fractional wavelet transforms on some function spaces

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In this paper, we give relationship between the continuous multilinear fractional wavelet transform and Fourier transform. Lastly, we consider boundedness of the continuous multilinear fractional wavelet transform on weighted Lebesgue spaces.

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KEYWORDS: Multilinear fractional wavelet transform, multilinear fractional Fourier transform, Weighted Lebesgue spaces.

Introduction

In this paper, the spaces $S(\mathbb{R})$ denotes as the Schwartz class of functions on $\mathbb{R}$, i.e., infinitely differentiable copmlex-valued functions, rapidly decreasing at infinity [2]. For $1 \leq p < \infty$, the spaces $L^p(\mathbb{R})$ denotes the usual Lebesgue space 1001[10]. Let $(X, \|.\|_X)$ be Banach algebra i.e., complete normed space, algebra, and $\|x \cdot y\|_X \leq \|x\|_X \|y\|_X$ for all $x, y \in X$. A Banach space $(B, \|.\|_B)$ is called Banach module X if B is a module over X in the algebraic sense for some multiplication, $(u, v) \longrightarrow u{\cdot}v$, and satisfies $\|u \cdot v\|_B \leq \|u\|_X \|v\|_B$ for all $u \in X$, $v \in B$. If the function ω is positive reel valued, measurable and locally bounded on $\mathbb{R}$ and satisfies the inequalities $\omega(x) \geq 1$, $\omega(x+y) \leq \omega(x)\,\omega(y)$ for all $x, y \in \mathbb{R}$ [10]. The weight function $\omega(x) = (1+|x|)^a$ is said polynominal type such that $\omega \leq v$. For $1 \leq p < \infty$, the weighted Lebesgue space is defined by $L^p_w(\mathbb{R}) = \{f : fw \in L^p(\mathbb{R})\}$ [10]. $L^p_w(\mathbb{R})$ is a vector space, where

$$\|f\|_{p,w} = \left\{ \int\limits_{\mathbb{R}^n} |f(x)|^p \, w(x)^p \, dx \right\}^{\frac{1}{p}}$$

and it is a Banach space according to the norm [10]. The multilinear mother wavelet is defined by:

$$\psi_{b,a}(t) = \prod_{j=1}^{n} T_{b_j} \psi_j^{a_j}(t_j) = |a|_n^{-\frac{1}{2}} \prod_{j=1}^{n} \psi_j\left(\frac{t_j - b_j}{a_j}\right),$$

for all $t = (t_1, ..., t_n) \in \mathbb{R}^n$ (*cf.* [3]). The multilinear fractional mother wavelet with multi-angle $\theta = (\theta, ..., \theta_n)$ is defined by:

$$\psi_{b,a,\theta}(t) = |a|_n^{-\frac{1}{2}} \prod_{j=1}^{n} \psi_j\left(\frac{t_j - b_j}{a_j}\right) e^{-\frac{i}{2}\left(t_j^2 - b_j^2\right)\cot\theta_j},$$

where $b = (b_1, ..., b_n)$, $t = (t_1, ..., t_n) \in \mathbb{R}^n$ and $a = (a_1, ..., a_n) \in \mathbb{R}^n_+$ [4]. We will use the notation $|a|_n = a_1 a_2 ... a_n$. Let $\mathbf{f} = (f_1, ..., f_n)$, $\mathbf{g} = (g_1, ..., g_n) \in S(\mathbb{R})^n =$

$S(\mathbb{R}) \times ... \times S(\mathbb{R})$. For all $t = (t_1, ..., t_n) \in \mathbb{R}^n$, the multilinear fractional convolution of $\mathbf{f}$ and $\mathbf{g}$ is defined as:

$$(\mathbf{f} *_\theta \mathbf{g})(t) = \prod_{j=1}^{n} \left(f_j *_{\theta_j} g_j\right)(t_j), \tag{1.1}$$

where $*_{\theta_j}$, $(j = 1, ..., n)$ denotes the fractional convolution $($*cf.* [3]$)$.
Let $\mathbf{f} = (f_1, ..., f_n), \psi = (\psi_1, ..., \psi_n) \in S(\mathbb{R})^n$. Assume that $\psi_{b,a,\theta}$ is a multilinear fractional mother wavelet. The multilinear continuous fractional wavelet transform of $\mathbf{f}$ with a multi-angle θ is defined by:

$$\begin{aligned} W_\psi^\theta \mathbf{f}(b,a) &= \int_{\mathbb{R}^n} \mathbf{f}(t) \overline{\psi_{b,a,\theta}(t)} dt = \langle \mathbf{f}, \psi_{b,a,\theta} \rangle \\ &= |a|_n^{-\frac{1}{2}} \int_{\mathbb{R}^n} \mathbf{f}(t) \prod_{j=1}^{n} \overline{\psi_j \left(\frac{t_j - b_j}{a_j}\right)} e^{-\frac{i}{2}\left(t_j^2 - b_j^2\right)\cot\theta_j} dt, \end{aligned}$$

for all $(b,a) \in \mathbb{R}^n \times \mathbb{R}_+^n$ $($*cf.* [3]$)$. Let $\psi = (\psi_1, ..., \psi_n) \in S(\mathbb{R})^n$. Then,

$$W_\psi^\theta \mathbf{f}(b,a) = e^{-\frac{i}{2}b^2\cot\theta} \prod_{j=1}^{n} \left(e^{\frac{i}{2}(.)^2\cot\theta_j} f_j * \psi_j^{a_j}\right)(b_j),$$

and

$$W_\psi^\theta \mathbf{f}(b,a) = \prod_{j=1}^{n} \left(f_j *_{\theta j} \psi_j^{a_j}\right)(b_j) = (\mathbf{f} *_\theta \psi^a)(b),$$

holds for all $\mathbf{f} = (f_1, ..., f_n) \in L^p(\mathbb{R})^n$ [4]. Let $\mathbf{f} = (f_1, ..., f_n) \in S(\mathbb{R})^n$. The multilinear continuous fractional Fourier transform of $\mathbf{f}$ with a multi-angle θ is defined as:

$$F^\theta \mathbf{f}(w) = \int_{\mathbb{R}^n} \mathbf{f}(t) \mathbf{K}^\theta(t,w)\, dt = \int_{\mathbb{R}^n} \mathbf{f}(t) \prod_{j=1}^{n} \mathbf{K}_j^{\theta_j}(t_j, w_j)\, dt, \tag{1.2}$$

where $\mathbf{K}_j^{\theta_j}(t_j, w_j)$ is kernel given by (1.2) and $C^\theta = (2\pi i \sin\theta)^{-\frac{1}{2}} e^{\frac{i\theta}{2}}$ [4]. Let $\mathbf{f} = (f_1, ..., f_n), \mathbf{g} = (g_1, ..., g_n) \in L^2(\mathbb{R})^n$. If $\psi = (\psi_1, ..., \psi_n), \varphi = (\varphi_1, ..., \varphi_n)$ are multi- wavelets in $S(\mathbb{R})^n$, then it' s written that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}_+^n} W_{\mathbf{\Psi}}^\theta(b,a) \overline{W_\varphi^\theta(b,a)} \frac{dbda}{|a|_n^2} = (2\pi)^n \prod_{j=1}^{n} \sin\theta_j C_{\psi_{j,\theta_j}} \langle f_j, g_j \rangle,$$

where

$$C_{\psi_{j,\theta_j}} = \int_{\mathbb{R}_+} \overline{F^{\theta_j}\left(e^{-\frac{i}{2}(.)^2\cot\theta_j}\psi_j\right)}(a_j)\, F^{\theta_j}\left(e^{-\frac{i}{2}(.)^2\cot\theta_j}\varphi_j\right)(a_j)\, a_j^{-1} da_j < \infty$$

for $(j = 1, ..., n)$ $($*cf.* [3]$)$. Let $\mathbf{f} = (f_1, ..., f_n) \in L^2(\mathbb{R})^n$. If $\psi = (\psi_1, ..., \psi_n)$ is multi-wavelet in $S(\mathbb{R})^n$, then $\mathbf{f}$ can be reconstructed by the following equation:

$$\mathbf{f}(t) = (2\pi)^{-n} \prod_{j=1}^{n} \frac{1}{\sin\theta_j C_{\psi_j,\theta_j}} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^n} W_\psi^\theta \mathbf{f}(b,a)\, \psi_{b,a,\theta}(t) \frac{dbda}{|a|_n^2}$$

$($*cf.* [3]$)$.

In the literature, Many researchers have done research on wavelet transform and fractional wavelet transform [1, 4, 5, 9, 6, 7, 8]. On some function spaces, some properties of the fractional wavelet transform and the ftactional Fourier transform were adapted to a multilinear form. So multilinear fractional transforms will make possible investigate the signal representations and local structures of signals in the n-dimensional fractional space.

Main results

In this section, firstly, we will give relationship between the continuous fractional multilinear wavelet transform and Fourier transform.

Theorem 1. *Let* $\mathbf{f} = (f_1, ..., f_n) \in L^2(\mathbb{R})^n$ *and let* $\varphi = (\varphi_1, ...\varphi_n)$ *be a multi-wavelet in* $S(\mathbb{R})^n$. *Then multilinear fractional Fourier transform is given by*

$$F^{\theta}\mathbf{f}(w) = (2\pi)^{-n} \prod_{j=1}^{n} \frac{1}{\sin\theta_j C_{\psi_j,\theta_j}} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n_+} W^{\theta}_{\boldsymbol{\Psi}}(b,a)\, F^{\theta}\psi_{b,a,\theta}(w) \frac{dbda}{|a|_n^2}.$$

Proof. Take any $f \in S(\mathbb{R})^n$. Then we have

$$\begin{aligned} F^{\theta}\mathbf{f}(w) &= \int\limits_{\mathbb{R}^n} \mathbf{f}(t) K^{\theta}(t,w)\, dt \\ &= \int\limits_{\mathbb{R}^n} \mathbf{f}(t) \prod_{j=1}^{n} K_j^{\theta_j}(t_j, w_j)\, dt \\ &= \int\limits_{\mathbb{R}} ... \int\limits_{\mathbb{R}} f_1(t_1) ... f_n(t_n) K_1^{\theta_1}(t_1,w_1) ... K_n^{\theta_n}(t_n,w_n)\, dt_1 ... dt_n \\ &= \left(\int\limits_{\mathbb{R}} f_1(t_1) K_1^{\theta_1}(t_1,w_1)\, dt_1 \right) ... \left(\int\limits_{\mathbb{R}} f_n(t_n) K_n^{\theta_n}(t_n,w_n)\, dt_n \right) \\ &= F^{\theta_1} f_1(w_1) ... F^{\theta_n} f_n(w_n) \\ &= \prod_{j=1}^{n} F^{\theta_j} f_j(w_j)\ . \end{aligned} \tag{1}$$

On the other hand, it's known that

$$F^{\theta_j} f_j(w_j) = (2\pi)^{-1} \frac{1}{\sin\theta_j C_{\psi_j,\theta_j}} \int\limits_{\mathbb{R}} \int\limits_{0}^{\infty} W^{\theta_j}_{\psi_j} f_j(b_j,a_j)\, F^{\theta_j}\psi_j^{b_j,a_j,\theta_j}(w_j) \frac{db_j da_j}{a_j^2} \tag{2.2}$$

for $(j = 1, ..., n)$ (*cf.* [9]). Using the equalities (2.1) and (2.2), we achieve

$$\begin{aligned} F^{\theta}\mathbf{f}(w) &= \prod_{j=1}^{n} F^{\theta_j} f_j(w_j) \\ &= F^{\theta_1} f_1(w_1) ... F^{\theta_n} f_n(w_n) \end{aligned}$$

$$= (2\pi)^{-1} \frac{1}{\sin\theta_1 C_{\psi_1,\theta_1}} \int\limits_{\mathbb{R}} \int\limits_{0}^{\infty} W^{\theta_1}_{\psi_1} f_1(b_1,a_1)\, F^{\theta_1}\psi_1^{b_1,a_1,\theta_1}(w_1) \frac{db_1 da_1}{a_1^2} \times$$

$$... \times (2\pi)^{-1} \frac{1}{\sin\theta_n C_{\psi_n,\theta_n}} \int\limits_{\mathbb{R}} \int\limits_0^\infty W_{\psi_n}^{\theta_n} f_n(b_n, a_n) F^{\theta_n} \psi_n^{b_n,a_n,\theta_n}(w_n) \frac{db_n da_n}{a_n^2}$$

$$= (2\pi)^{-1} \prod_{j=1}^n \frac{1}{\sin\theta_j C_{\psi_j,\theta_j}} \int\limits_{\mathbb{R}} ... \int\limits_{\mathbb{R}} \int\limits_0^\infty ... \int\limits_0^\infty W_{\psi_1}^{\theta_1} f_1(b_1, a_1) ... W_{\psi_n}^{\theta_n} f_n(b_n, a_n)$$

$$\times F^{\theta_1} \psi_1^{b_1,a_1,\theta_1}(w_1) ... F^{\theta_n} \psi_n^{b_n,a_n,\theta_n}(w_n) db_1 ... db_n \frac{da_1}{a_1^2} ... \frac{da_n}{a_n^2}$$

$$= (2\pi)^{-n} \prod_{j=1}^n \frac{1}{\sin\theta_j C_{\psi_j,\theta_j}} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}_+^n} \left(\prod_{j=1}^n W_{\psi_j}^{\theta_j} f_j(b_j, a_j) \right) \prod_{j=1}^n F^{\theta_j} \psi_j^{b_j,a_j,\theta_j}(w_j) \frac{dbda}{|a|_n^2}$$

$$= (2\pi)^{-n} \prod_{j=1}^n \frac{1}{\sin\theta_j C_{\psi_{j},\theta_j}} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}_+^n} W_{\mathbf{\Psi}}^{\theta}(b, a) F^{\theta} \psi_{b,a,\theta}(w) \frac{dbda}{|a|_n^2}.$$

□

Theorem 2. *Let* $w_{j,}\,(j = 1, ..., n)$ *be polynominal type weight function and let be* $w(b) = \prod\limits_{j=1}^n w_j(b_j)$. *Suppose that* $\mathbf{\Psi} = (\Psi_1, ...\Psi_n) \in S(\mathbb{R})^n$ *is multi-wavelet. Then,*

$$\left\| \left(W_{\mathbf{\Psi}}^{\theta} \mathbf{f} \right)(., a) \right\|_{L_w^p(\mathbb{R}^n)} \leq |a|_n^{\frac{1}{2}} w(a) \prod_{j=1}^n \|f_j\|_{L_{w_j}^p(\mathbb{R})} \prod_{j=1}^n \|\Psi_j\|_{L_{w_j}^1(\mathbb{R})}$$

holds for all $\mathbf{f} = (f_1, ..., f_n) \in L_w^p(\mathbb{R})^n$ *and for any* $a \in \mathbb{R}_+^n$.

Proof. Let $a \in \mathbb{R}_+^n$. If we use Theorem 1 in [3], we achieve

$$\begin{aligned} \left\| \left(W_{\mathbf{\Psi}}^{\theta} \mathbf{f} \right)(., a) \right\|_{L_w^p(\mathbb{R}^n)} &= \left\| e^{-\frac{i}{2} b^2 \cot\theta} \prod_{j=1}^n \left(e^{\frac{i}{2}(.)^2 \cot\theta_j} f_j * \Psi_j^{a_j} \right) \right\|_{L_w^p(\mathbb{R}^n)}^p \\ &= \left\| \prod_{j=1}^n \left(e^{\frac{i}{2}(.)^2 \cot\theta_j} f_j * \Psi_j^{a_j} \right) \right\|_{L_w^p(\mathbb{R}^n)}^p \\ &= \int\limits_{\mathbb{R}^n} \left| \prod_{j=1}^n \left(e^{\frac{i}{2}(.)^2 \cot\theta_j} f_j * \Psi_j^{a_j} \right) \right|^p w(b)\, db \end{aligned}$$

$$= \int\limits_{\mathbb{R}} \dots \int\limits_{\mathbb{R}} \left| \prod_{j=1}^n \left(e^{\frac{i}{2}(.)^2 \cot\theta_j} f_j * \Psi_j^{a_j} \right) \right|^p w_1(b_1) \dots w_n(b_n)\, db_1 \dots db_n$$

$$= \int\limits_{\mathbb{R}} \dots \int\limits_{\mathbb{R}} \left| \left(e^{\frac{i}{2}(.)^2 \cot\theta_1} f_1 * \Psi_1^{a_1} \right) \right|^p \dots \left| \left(e^{\frac{i}{2}(.)^2 \cot\theta_n} f_n * \Psi_n^{a_n} \right) \right|^p w_1(b_1) \dots w_n(b_n)\, db_1 \dots db_n$$

$$= \left(\int\limits_{\mathbb{R}} \left| \left(e^{\frac{i}{2}(.)^2 \cot\theta_1} f_1 * \Psi_1^{a_1} \right) \right|^p w_1(b_1)\, db_1 \right) \dots \left(\int\limits_{\mathbb{R}} \left| \left(e^{\frac{i}{2}(.)^2 \cot\theta_n} f_n * \Psi_n^{a_n} \right) \right|^p w_n(b_n)\, db_n \right)$$

$$= \left\| e^{\frac{i}{2}(.)^2 \cot\theta_1} f_1 * \Psi_1^{a_1} \right\|_{L_{w_1}^p(\mathbb{R})}^p \dots \left\| e^{\frac{i}{2}(.)^2 \cot\theta_n} f_n * \Psi_n^{a_n} \right\|_{L_{w_n}^p(\mathbb{R})}^p$$

$$= \left\| \prod_{j=1}^{n} \left(e^{\frac{i}{2}(.)^2 \cot\theta_j} f_j * \Psi_j^{a_j} \right) \right\|_{L^p_{w_j}(\mathbb{R})}^p .$$

We know that $L^p_{w_j}(\mathbb{R})$ is Banach module over $L^1_{w_j}(\mathbb{R})$, $(j = 1, ..., n)$. So using last equality,

$$\begin{aligned}
\left\| \left(W_{\Psi}^{\theta} \mathbf{f} \right)(.,a) \right\|_{L^p_w(\mathbb{R}^n)} &= \prod_{j=1}^{n} \left\| \left(e^{\frac{i}{2}(.)^2 \cot\theta_j} f_j * \Psi_j^{a_j} \right) \right\|_{L^p_{w_j}(\mathbb{R})}^p \\
&\leq \prod_{j=1}^{n} \left\| e^{\frac{i}{2}(.)^2 \cot\theta_j} f_j \right\|_{L^p_{w_j}(\mathbb{R})}^p \left\| \Psi_j^{a_j} \right\|_{L^1_{w_j}(\mathbb{R})}^p \\
&= \prod_{j=1}^{n} \| f_j \|_{L^p_{w_j}(\mathbb{R})}^p \prod_{j=1}^{n} \left\| \Psi_j^{a_j} \right\|_{L^1_{w_j}(\mathbb{R})}^p . \qquad (2.3)
\end{aligned}$$

Using that w_j is polynominal type and changing variable $\frac{x}{a_j} = u_j$, $(j = 1, ..., n)$, then we have

$$\begin{aligned}
\left\| \Psi_j^{a_j} \right\|_{L^1_{w_j}(\mathbb{R})}^p &= \int_{\mathbb{R}} \left| \Psi_j^{a_j}(x) \right| w_n dx = \int_{\mathbb{R}} a_j^{-\frac{1}{2}} \left| \Psi_j\left(\frac{x}{a_j} \right) \right| w_j(x) \, dx \\
&= \int_{\mathbb{R}} a_j^{-\frac{1}{2}} \left| \Psi_j(u_j) \right| w_j(a_j u_j) \, a_j du_j \\
&= a_j^{\frac{1}{2}} \int_{\mathbb{R}} \left| \Psi_j(u_j) \right| w_j(a_j u_j) \, du_j \\
&\leq a_j^{\frac{1}{2}} \int_{\mathbb{R}} \left| \Psi_j(u_j) \right| w_j(a_j) \, w_j(u_j) \, du_j \\
&= a_j^{\frac{1}{2}} w_j(a_j) \int_{\mathbb{R}} \left| \Psi_j(u_j) \right| w_j(u_j) \, du_j \\
&= a_j^{\frac{1}{2}} w_j(a_j) \left\| \Psi_j^{a_j} \right\|_{L^1_{w_j}(\mathbb{R})} . \qquad (2)
\end{aligned}$$

Combing (2.3) and (2.4), we obtain

$$\begin{aligned}
\left\| \left(W_{\Psi}^{\theta} \mathbf{f} \right)(.,a) \right\|_{L^p_w(\mathbb{R}^n)} &\leq \prod_{j=1}^{n} \| f_j \|_{L^p_{w_j}(\mathbb{R})}^p \prod_{j=1}^{n} a_j^{\frac{1}{2}} w_j(a_j) \left\| \Psi_j^{a_j} \right\|_{L^1_{w_j}(\mathbb{R})}^p \\
&= \prod_{j=1}^{n} a_j^{\frac{1}{2}} w_j(a_j) \prod_{j=1}^{n} \| f_j \|_{L^p_{w_j}(\mathbb{R})}^p \prod_{j=1}^{n} \left\| \Psi_j^{a_j} \right\|_{L^1_{w_j}(\mathbb{R})}^p \\
&= \prod_{j=1}^{n} a_j^{\frac{1}{2}} \prod_{j=1}^{n} w_j(a_j) \prod_{j=1}^{n} \| f_j \|_{L^p_{w_j}(\mathbb{R})}^p \prod_{j=1}^{n} \left\| \Psi_j^{a_j} \right\|_{L^1_{w_j}(\mathbb{R})}^p \\
&= |a|_n^{\frac{1}{2}} w(a) \prod_{j=1}^{n} \| f_j \|_{L^p_{w_j}(\mathbb{R})} \prod_{j=1}^{n} \| \Psi_j \|_{L^1_{w_j}(\mathbb{R})} .
\end{aligned}$$

□

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References

[1] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF, SIAM: Philadelphia, PA, USA, 1992.

[2] C. Gasquet and P. Witomski, *Fourier analysis and applications, filtering, numerial computation, wavelets*, Springer, New York, NY, USA, 1998.

[3] Ö. Kulak, *On the multilinear fractional transforms*, Symmetry, **13 (5)**, 740, 2021.

[4] Ö. Kulak and I. Aydın, *Inverse continuous wavelet transform in weighted variable exponent amalgam spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69**, 1171–1183, 2020.

[5] R. S. Pathak, A. Prasad and M. Kumar, *Fractional Fourier transform of tempered distributions and generalized pseudo-differential operator*, J. Pseudo-Differ. Oper. Appl. **3**, 239–254, 2012.

[6] A. Prasad and P. Kumar, *Fractional wavelet transform in terms of fractional convolution*, Progr. Fract. Differ. Appl. **3**, 201–210, 2015.

[7] A. Prasad and P. Kumar, *The continuous fractional wavelet transform on a generalized sobolev space*, Int. J. Wavel. Multiresolut. Inf. Proces. **14**, 2016; Article ID: 1650046.

[8] A. Prasad and P. Kumar, *Fractional wavelet transform on some function spaces*, Proc. Natl. Sci. India Sect. A Phys. Sci. **86**, 57–64, 2016.

[9] A. Prasad, S. Manna, A. Mahato and V.K. Singh, *The generalized continuous wavelet transform associated with the fractional Fourier transform*, J. Comput. Appl. Math. **259**, 660–671, 2014.

[10] H. Reiter, *Classical harmonic analysis and locally compact group*, Oxford Universty Pres, Oxford, 1968.

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Observations on computational formula of the Bernoulli numbers and combinatorial numbers and polynomials via determinant method

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Computational methods that can be used to create mathematical models for real-world problems are crucial in almost all interdisciplinary fields spanning mathematics, science and engineering. The purpose of this presentation is to examine and investigate computational formulas for special numbers and polynomials by blending some new and old methods. With the help of the determinant method, computational formulas of the Bernoulli numbers, as well as formulas involving a certain family of combinatorial numbers and polynomials, and open questions will be brought to light.

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Introduction

Computational methods and computational formulas have been many applications not only in pure and applied mathematics involving , but also in the physical, biological, social, and behavioral sciences, and computational nature focusing on analysis of algorithms, and numerical analysis. Throughout, of this survey paper, we use notations and some formulas book of Pap [11]. In this book, Pap gave determinat method in order to evaluate numerical values of the Bernoulli numbers.

Setting

$$\frac{\sum_{n=0}^{\infty}\alpha_n(z-a)^n}{\sum_{n=0}^{\infty}\beta_n(z-a)^n}=\sum_{n=0}^{\infty}\delta_n(z-a)^n, \tag{1}$$

where $|z-a|<R$, $R>0$ and $\beta_0\neq 0$ (*cf.* [11, p.148]).

How can one express an explicit formula for the coefficients δ_n? By using (1), one has

$$\sum_{n=0}^{\infty}\alpha_n(z-a)^n=\left(\sum_{n=0}^{\infty}\delta_n(z-a)^n\right)\left(\sum_{n=0}^{\infty}\beta_n(z-a)^n\right).$$

By using the Cauchy product rules with the uniqueness of the expansion in power series in the disc $\{z|\ |z-a|<R\}$ yields

$$\sum_{n=0}^{\infty}\alpha_n(z-a)^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\delta_k\beta_{n-k}(z-a)^n.$$

It is known from the inspiration of the equality of the power series that since the coefficients in the above equation $(z-a)^n$ are equal, the following results can be easily calculated:

$$\begin{aligned}
\alpha_0 &= \delta_0\beta_0, \\
\alpha_1 &= \delta_0\beta_1+\delta_1\beta_0, \\
\alpha_2 &= \delta_0\beta_2+\delta_1\beta_1+\delta_2\beta_0, \\
&\vdots \\
\alpha_n &= \delta_0\beta_n+\delta_1\beta_{n-1}+\delta_2\beta_{n-2}+\ldots+\delta_n\beta_0 \\
&\vdots
\end{aligned} \tag{2}$$

The above table gives us an infinite system of linear equations with unknown $\delta_0, \delta_1, \delta_2, \ldots, \delta_n, \ldots$. It can be easily observed that the system of linear equations mentioned above has a special structure. Because for every value of n, the first $n+1$ equations include only the first $n+1$ unknown δ_k.

As a result, the solution according to the well-known determinant method can be briefly given as follows:

$$\delta_n = \frac{1}{\beta_0^{n+1}}\begin{vmatrix} \beta_0 & 0 & 0 & 0 & \alpha_0 \\ \beta_1 & \beta_0 & 0 & 0 & \alpha_1 \\ \beta_2 & \beta_1 & \beta_0 & 0 & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_{n-1} & \beta_{n-2} & \ldots & \alpha_n \end{vmatrix}, \tag{3}$$

where

$$\begin{vmatrix} \beta_0 & 0 & 0 & 0 & \alpha_0 \\ \beta_1 & \beta_0 & 0 & 0 & \alpha_1 \\ \beta_2 & \beta_1 & \beta_0 & 0 & \alpha_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_{n-1} & \beta_{n-2} & \ldots & \alpha_n \end{vmatrix} = \beta_0^{n+1} \neq 0$$

(*cf.* [11, p.149]).

For the above method, we give the following well-known example for the Bernoulli numbers, which are defined by means of the following generating function:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

The function $\frac{t}{e^t-1}$ can be written as power series of the fraction of two analytic functions. That is

$$\frac{t}{\sum_{n=1}^{\infty} \frac{1}{n!} t^n} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

Therefore

$$t = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n\right)\left(\sum_{n=1}^{\infty} \frac{1}{n!} t^n\right),$$

where $B_0 = 1$ and $\frac{1}{1!} \neq 0$. Then combining the above equation with (2) and (3) yields the following computational formula for the Bernoulli numbers, with the aid of determinant method:

$$\delta_n = \frac{(-1)^n}{\alpha_0^n}\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \dots & \frac{1}{2!} \end{vmatrix} = \frac{B_n}{n!}, \tag{4}$$

where $\delta_0 = 1$ and B_n denotes the Bernoulli numbers.

The formula in (4) was proofed by Pap [11, p.149] in 1999. Unfortunately, the same proof given by Chen [3] in 2003.

By using (1), Gradshteyn and Ryzhik [4, p. 17] gave another determinant method. That is

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0}\sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_n + \frac{1}{a_0}\sum_{k=1}^{n} c_{n-k} a_k - b_n = 0$$

or

$$c_n = \frac{(-1)^n}{a_0^n}\begin{vmatrix} a_1b_0 - a_0b_1 & a_0 & 0 & \dots & 0 \\ a_2b_0 - a_0b_2 & a_1 & a_0 & \dots & 0 \\ a_3b_0 - a_0b_3 & a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1}b_0 - a_0b_{n-1} & a_{n-2} & a_{n-3} & \ddots & a_0 \\ a_nb_0 - a_0b_n & a_{n-1} & a_{n-2} & \dots & a_1 \end{vmatrix}$$

see also [18, p. 693].

Further remarks and observations

In [14], the second author constructed the following generating functions:

$$\frac{2}{\lambda(1+\lambda t) - 1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!}$$

and

$$\frac{2(1+\lambda t)^x}{\lambda(1+\lambda t) - 1} = \sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!}.$$

In [10], Kucukoglu and Simsek, generalized the polynomials $Y_n(x;\lambda)$ and the number $Y_n(\lambda)$. After that, Khan *et al.* [6]- [5] constructed the 2-variable Simsek polynomials, which are given as follows:

$$\frac{2(1+\lambda t)^x(1+\beta t^2)^y}{\lambda(1+\lambda t) - 1} = \sum_{n=0}^{\infty} Y_n(x,y;\lambda,\beta)\frac{t^n}{n!}.$$

They investigated some properties of these polynomials. Thus they gave relation among the polynomials $Y_n(x,y;\lambda,\beta)$, the Changhee polynomials, the Daehee polynomials, and also the Cauchy numbers, the Changhee numbers, the Daehee numbers, and the Simsek numbers. They also show that the polynomials $Y_n(x,y;\lambda,\beta)$ can be represented by hypergeometric function. The graphical representations of the 2-variable Simsek polynomials were given for appropriate values of the index n and parameters λ and β. They also found the derivation of non-linear differential equation and related identities for the Simsek numbers.

The motivation of this paper is find generalizaton of the 2-variable Simsek polynomials of order $w \in \mathbb{C}$, set of complex numbers.

In [8], Kucukoglu constructed the following generating functions for the several variables multiparametric Hermite-based Simsek polynomials:

$$\frac{e^{zt+y_1t+y_2t^2+\cdots+y_mt^m}2^k\prod\limits_{j=1}^{k}\left(1+\omega_jt^j\right)^{x_j}}{\prod\limits_{j=1}^{k}\left(\omega_j\left(1+\omega_jt\right)-1\right)}=\sum_{n=0}^{\infty}{}_HY_n^{(k)}\left(\overrightarrow{x},\overrightarrow{y},z,\overrightarrow{\omega},m,k\right)\frac{t^n}{n!}, \tag{5}$$

($cf.$ [8]).

By using the above equation (5), we define the following generating function for the Simsek polynomials of complex number order:

$$\frac{2^z(1+\lambda t)^x}{(\lambda^2t+\lambda-1)^z}=\sum_{n=0}^{\infty}Y_n^{(z)}(x;\lambda)\frac{t^n}{n!},$$

where $z \in \mathbb{C}$, $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ (or $\mathbb{R}$).

We investigate many properties of the above generating function and the polynomials $Y_n^{(z)}(x;\lambda)$ on complex z-plane.

Conclusion

In this work, we surveyed and researched computational formulas for certain family of special numbers and polynomials by using generating functions and determinant methods. Using determinant method, computational formulas of the Bernoulli numbers, as well as formulas involving a certain family of combinatorial numbers and polynomials were given. Some open questions were also given.

Our future project is to give another generalizations of the polynomials $Y_n(x;\lambda)$. Using these generalizations, we will study and research some computational formula for generalizations of the polynomials $Y_n(x;\lambda)$, and generalizations of the numbers $Y_n(\lambda)$ with aid of determinant method, as well as investigate application of the polynomials $Y_n^{(z)}(x;\lambda)$.

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References

[1] T. Arakawa, T. Ibukiyama and M. Kaneko, *Bernoulli numbers and zeta functions*, Springer, New York, 2014.

[2] A. Bayad, *Jacobi forms in two variables: Multiple elliptic Dedekind sums, the Kummer-Von Staudt Clausen congruences for elliptic Bernoulli functions and values of Hecke L-functions*, Montes Taurus J. Pure Appl. Math. **1 (2)**, 58–129, 2019; Article ID: MTJPAM-D-19-00005.

[3] H. Chen, *Bernoulli numbers via determinants*, International Journal of Mathematical Education in Science and Technology, **34 (2)**, 291–297, 2003; DOI:10.1080/0020739031000158335.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products* (Ed. by A. Jeffrey and D. Zwillinger), Academic Press, New York, 7th edition, 2007.

[5] S. Khan, T. Nahid and M. Riyasat, *Partial derivative formulas and identities involving 2-variable Simsek polynomials*, Bol. Soc. Mat. Mex. **26**, 1–13, 2020; https://doi.org/10.1007/s40590-019-00236-4.

[6] S. Khan, T. Nahid and M. Riyasat, *Properties and graphical representations of the 2-variable form of the Simsek polynomials*, Vietnam J. Math. **50**, 95–109, 2022; https://doi.org/10.1007/s10013-020-00472-6.

[7] I. Kucukoglu, *Identities for the multiparametric higher-order Hermite-based Peters-type Simsek polynomials of the first kind*, Montes Taurus J. Pure Appl. Math. **5 (1)**, 102–123, 2023; Article ID: MTJPAM-D-23-00002.

[8] I. Kucukoglu, *Generating functions for multiparametric Hermite-based Simsek numbers and polynomials in several variables*, To appear in Montes Taurus J. Pure Appl. Math. **5 (3)**, 2024; Article ID: MTJPAM-D-23-00027.

[9] I. Kucukoglu and Y. Simsek, *New formulas and numbers arising from analyzing combinatorial numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **3 (3)**, 238–259, 2021; Article ID: MTJPAM-D-20-00038.

[10] I. Kucukoglu, B. Simsek and Y. Simsek, *An approach to negative hypergeometric distribution by generatingfunction for special numbers and polynomials*, Turk. J. Math. **43 (5)**, 2337–2353, 2019.

[11] E. Pap, *Complex analysis through examples and exercises*, Springer Netherlands, 1999; DOI: 10.1007/978-94-017-1106-7.

[12] E. Polat and Y. Simsek, *Notes on Bernoulli polynomials associated with linear map on the set of rational numbers*, Proceedings Book of the 13th Symposium on Generating Functions of Special Numbers and Polynomials and their Applications (GFSNP), 167–169, 2023; ISBN: 978-625-00-1128-7.

[13] E. Polat and Y. Simsek, *Some certain families of special polynomials derive from linear matrix form of transformation the using of rational numbers*, Proceedings Book of the 6th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM), 159–161, 2023; ISBN: 978-2-491766-01-6.

[14] Y. Simsek, *Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q -integrals*, Turk. J. Math. **42 (2)**, 557–577, 2018.

[15] Y. Simsek, *Explicit formulas for p-adic integrals: Approach to p-adic distribution and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 1–76, 2019.

[16] Y. Simsek, *Interpolation functions for new classes special numbers and polynomials via applications of p-adic integrals and derivative operator*, Montes Taurus J. Pure Appl. Math. **3 (1)**, 38–61, 2021; Article ID: MTJPAM-D-20-00000.

[17] Y. Simsek, *Applications of constructed new families of generating-type functions interpolating new and known classes of polynomials and numbers*, Math. Meth. Appl. Sci. **44**, 11245–11268, 2021.

[18] Y. Simsek, *Some families of finite sums associated with interpolation functions for many classes of special numbers and polynomials and hypergeometric series*, In: Analysis, Geometry, Nonlinear Optimization and Applications (Ed. by P. M. Pardalos and T. M. Rassias), 677–750, 2023.

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Behavior of the generating function for combinatorial numbers under the difference operator

Elif Sukruoglu *1 *and Yilmaz Simsek* 2

The main purpose of this presentation is to investigate the behavior of the certain classes combinatorial numbers involving higher order binomial coefficients under the difference operator. By using applications of this operator, we derive some formulas and finite sums involving the certain classes combinatorial numbers such as the combinatorial numbers $y_6(m,n;\lambda,r)$, the combinatorial numbers $y_1(m,n;\lambda)$, the Daehee numbers, and the Changhee numbers.

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Keywords: Combinatorial numbers, Generating function, Difference operator

Introduction

The finite difference operator is roughly determined by the mathematical expression

$$T(d,e)\left[G(y)\right] = G\left(y+d\right) - G\left(y+e\right).$$

This operator is used primarily in the finite difference approximation of the derivative of a function, differential equations involving the numerical solution of boundary value problems, numerical calculations of functions, and the construction of special numbers including Stirling numbers.

When $e=1$, the operator $T(d,e)$ reduces to the Δ_d operator. That is,

$$T(d,1)\left[G(y)\right] = \Delta_d\left[G(y)\right] = G\left(y+d\right) - G\left(y\right).$$

(*cf.* [8, 20]).

The finite difference operator have been used and applied all most all branches of pure and applied sciences. Especially, this operator can be used to finde numerical evaluations and behaviour of the functions, mainly and polynomials. Special values of these beheviour gives us special numbers. For instance, the second kind Stirling numbers can be obtained by this way.

In this presentation, applying the operator

$$\Delta_1\left[G(y)\right] = \Delta\left[G(y)\right] = G\left(y+1\right) - G\left(y\right).$$

to the combinatorial numbers $y_6(m,n;\lambda,r)$, which are defined below, we derive some new identities and relations.

Successive differences can also be taken. For example

$$\begin{aligned}\Delta^2 G(x) &= \Delta[\Delta G(x)]\\ &= \Delta[G(x+1)-G(x)]\\ &= G(x+2)-2G(x+1)+G(x).\end{aligned}$$

We call Δ^2 the second order difference operator or difference operator of order 2. In general we define the nth order difference operator by

$$\begin{aligned}\Delta^n G(x) &= \Delta[\Delta^{n-1}G(x)] \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j G(x+n-j)\end{aligned} \tag{1}$$

(cf. [8], [20]).

The motivation of this work is to focus applications of the operator Δ to the following combinatorial numbers $y_6(m,n;\lambda,r)$, which were discovered the second author [14]:

$$F_{y_6}(t,n;\lambda,r) := \frac{1}{n!}\, {}_rF_{r-1}\left[\begin{array}{c} -n,-n,...,-n \\ 1,1,...,1 \end{array} ; (-1)^r \lambda e^t\right] \tag{2}$$

$$= \sum_{m=0}^{\infty} y_6(m,n;\lambda,r)\frac{t^m}{m!}, \tag{3}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $n,r \in \mathbb{N} = \{1,2,\dots\}$, $m \in \mathbb{N} \cup \{0\}$ and ${}_rF_{r-1}$ denotes the following generalized hypergeometric function:

$${}_pF_q\left[\begin{array}{c} \alpha_1,...,\alpha_p \\ \beta_1,...,\beta_q \end{array} ; z\right] = \sum_{m=0}^{\infty} \left(\frac{\prod_{j=1}^{p} (\alpha_j)_m}{\prod_{j=1}^{q} (\beta_j)_m}\right) \frac{z^m}{m!},$$

where $(\alpha)_m$ denotes the Pochhammer symbol defined by

$$(\alpha)_m = \alpha(\alpha+1)\dots(\alpha+m-1)$$

with $(\alpha)_0 = 1$, such that the above series converges for all z if $p < q+1$, and for $|z| < 1$ if $p = q+1$. For this series one can assumed that all parameters have real or complex values, except for the β_j, $j = 1,2,...,q$ none of which is equal to zero or to a negative integer (See, for details, [14, 19, 21]).

By using hypergeometric series techniques in (2), the function $F_{y_6}(t,n;\lambda,r)$ can also be written as follows:

$$\begin{aligned}F_{y_6}(t,n;\lambda,r) &= \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}^r \lambda^j e^{tj} \\ &= \sum_{m=0}^{\infty} y_6(m,n;\lambda,p)\frac{t^m}{m!}\end{aligned} \tag{4}$$

(*cf.* [14, p. 1329]).

The numbers

$$\frac{\partial^m}{\partial t^m}\{F_{y_6}(t,n;\lambda,r)\}|_{t=0} = y_6(m,n;\lambda,p) + \sum_{v=m+1}^{\infty} y_6(v,n;\lambda,p)\frac{t^v}{v!}\,|_{t=0}$$

From the above equation, we see that

$$y_6(m,n;\lambda,p) = \frac{\partial^m}{\partial t^m}\{F_{y_6}(t,n;\lambda,r)\}|_{t=0}$$

(*cf.* [14, p. 1329])

Conspicuously, the combination of (3) and (4) implies that the numbers $y_6(m,n;\lambda,r)$ can be expressed explicitly by the following finite sum:

$$y_6(m,n;\lambda,p) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p} k^m \lambda^k, \tag{5}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $n,p \in \mathbb{N} = \{1,2,\dots\}$ and $m \in \mathbb{N}\cup\{0\}$ (*cf.* [14, p. 1347]).

Recently, these numbers have been used by many researchers. Here noting that the numbers $y_6(m,n;\lambda,r)$ is referred *combinatorial Simsek numbers of the sixth kind* because of its index as in this study of Kucukoglu [7].

The most obvious relationship between the first kind and sixth kind combinatorial Simsek numbers is as follows:

$$y_6(m,n;\lambda,1) = y_1(m,n;\lambda)$$

(*cf.* [14, 13]).

Main results

In this section, we give some applications of the Δ^m to the generating function for the numbers $y_6(m,n;\lambda,r)$. Thus, we derive some formulas.

By applying (1) to (4), we get

$$\Delta_t^m[F_{y_6}(t,n;\lambda,r)] = \frac{1}{n!}\sum_{k=0}^{m}\sum_{j=0}^{n}(-1)^k\binom{n}{j}^{r}\binom{m}{k}\lambda^j e^{(t+m-k)j}$$

Therefore, we get the following difference equation:

$$\Delta_t^m[F_{y_6}(t,n;\lambda,r)] = \sum_{k=0}^{m}(-1)^k\binom{m}{k}F_{y_6}(t+m-k,n;\lambda,r). \tag{6}$$

Similarly, we get

$$\Delta_t^m[F_{y_6}(t,n;\lambda,r)] = \sum_{v=0}^{\infty}y_6(v,n;\lambda,p)\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(t+m-k)^v}{v!} \tag{7}$$

Combining (6) and (7) yields the following theorem:

Theorem 1. *Let $m \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{\infty}y_6(v,n;\lambda,p)\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{(t+m-k)^v}{v!} = \sum_{k=0}^{m}(-1)^k\binom{m}{k}F_{y_6}(t+m-k,n;\lambda,r). \tag{8}$$

Substituting $m=1$ into (8), we get

$$\sum_{v=0}^{\infty}y_6(v,n;\lambda,p)\frac{(t+1)^v - t^v}{v!} = F_{y_6}(t+1,n;\lambda,r) - F_{y_6}(t,n;\lambda,r).$$

On the other hand we have

$$\Delta^d[y_6(m,n;\lambda,p)] = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^{p}k^m\sum_{j=0}^{d}\binom{d}{j}(-1)^j(\lambda+d-j)^k. \tag{9}$$

Therefore

$$\Delta^d [y_6(m,n;\lambda,p)] = \sum_{j=0}^{d} \binom{d}{j} (-1)^j y_6(m,n;\lambda+d-j,p).$$

Putting $\lambda = -v$ in (9), we obtain

$$\Delta^d [y_6(m,n;-v,p)] = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}^p k^m \sum_{j=0}^{d} \binom{d}{j} (-1)^j (-j)^k. \tag{10}$$

Combining the above equation with following numbers, which were defined by the second author [13]:

$$y_1(m,k;\lambda) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^m \lambda^j,$$

we arrive at the following theorem:

Theorem 2. *Let $m \in \mathbb{N}_0$. Then we have*

$$\Delta^d [y_6(m,n;-v,p)] = \frac{d!}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k}^p k^m y_1(k,d;-1). \tag{11}$$

Substituting the Daehee numbers: $(d+1)\,D_d = (-1)^d d!$ (cf. [5]) into (11), we get the following corollary:

Corollary 3. *Let $m \in \mathbb{N}_0$. Then we have*

$$\Delta^d [y_6(m,n;-v,p)] = \frac{(d+1)\,D_d}{n!} \sum_{k=0}^{n} (-1)^{k+d} \binom{n}{k}^p k^m y_1(k,d;-1). \tag{12}$$

Conclusion

We gave some properties of the difference operator. The purpose of this study is to give applications of the difference operator to the certain classes combinatorial numbers involving higher order binomial coefficients, we found some formulas and finite sums involving the certain classes combinatorial numbers such as the combinatorial numbers $y_6(m,n;\lambda,r)$, the combinatorial numbers $y_1(m,n;\lambda)$, the Daehee numbers, and the Changhee numbers.

Our future project will be to investigate other properties of this operator and its applications to othe classes of special polynomials.

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References

[1] R. Golombek, D. Marburg, *Aufgabe 1088, Summen mit quadraten von Binomialkoeffizienten*, El. Math. **50**, 125–131, 1995.

[2] M. Goubi, *An affirmative answer to two questions concerning special case of Simsek numbers and open problems*, Appl. Anal. Discrete Math. **14**, 94–105, 2020.

[3] M. Goubi, *Generating functions for generalization Simsek numbers and their applications*, Appl. Anal. Discrete Math. **17**, 262–272, 2023.

[4] N. Kilar, *Asymptotic expressions and formulas for finite sums of powers of Binomial coefficients involving special numbers and polynomials*, J. Inequal. Spec. Funct. **14 (1)**, 51–67, 2023.

[5] D. S. Kim and T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. (Ruse) **7 (120)**, 5969-5976, 2013.

[6] D. S. Kim, T. Kim and J. Seo, *A note on Changhee numbers and polynomials*, Adv. Stud. Theor. Phys. **7**, 993–1003, 2013.

[7] I. Kucukoglu, *Generating functions for Divisor sums and totative sums arising from combinatorial Simsek numbers*, Filomat **38 (5)**, 1513–1529, 2024.

[8] M. Thomson and L. Melville, *The calculus of finite differences*, American Mathematical Soc., 2000.

[9] V. H. Moll, *Numbers and Functions: from a Classical-experimental Mathematician's Point of View*, Student Mathematical Library (Volume 65), American Mathematical Society, Providence, Rhode Island, 2012.

[10] S. Roman, *The Euler polynomials. §4.2.3 in the umbral calculus*, Academic Press, New York, 100–106, 1984.

[11] Y. Simsek, *Computation methods for combinatorial sums and Euler-type numbers related to new families of numbers*, Math. Methods Appl. Sci. **40 (7)**, 2347–2361, 2017.

[12] Y. Simsek, *Identities and relations related to combinatorial numbers and polynomials*, Proc. Jangjeon Math. Soc. **20 (1)**, 127–135, 2017.

[13] Y. Simsek, *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials*, Appl. Anal. Discrete Math. **12 (1)**, 1–35, 2018.

[14] Y. Simsek, *Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers*, J. Math. Anal. Appl. **477 (2)**, 1328–1352, 2019.

[15] Y. Simsek, *Some new families of special polynomials and numbers associated with finite operators*, Symmetry **12 (2)**, 2020; Article ID: 237, DOI: 10.3390/sym12020237.

[16] Y. Simsek, *On Boole-type combinatorial numbers and polynomials*, Filomat **34 (2)**, 559–565, 2020.

[17] Y. Simsek, *Relations between a certain combinatorial numbers and Fibonacci numbers*, Proceedings Book of MICOPAM 2020-2021, ISBN: 978-625-00-0397-8; pp. 23–25, 2021.

[18] Y. Simsek, *Some classes of finite sums related to the generalized Harmonic functions and special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **4 (3)**, 61–79, 2022.

[19] Y. Simsek, *Some families of finite sums associated with interpolation functions for many classes of special numbers and polynomials and Hypergeometric series*, In: Analysis, Geometry, Nonlinear Optimization and Applications, (Ed. by P. M. Pardalos, T. M. Rassias), World Scientific Publishing Co. Pte. Ltd., pp.677-750, 2023, https://doi.org/10.1142/9789811261572_0026

[20] R. M. Spiegel, *Calculus of finite differences and difference equations: Incl. 420 solved problems*, McGraw-Hill, 1971.

[21] H. M. Srivastava and J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[22] A. Xu, *On an open problem of Simsek concerning the computation of a family of special numbers*, Appl. Anal. Discrete Math. **13 (1)**, 061–072, 2019.

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On the pedal and contrapedal curves of spacelike framed immersions

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In this work, we obtain Frenet type formulae of spacelike framed immersions in Minkowski 3-space and describe their pedal and contrapedal curves. We also examine the necessary conditions for these curves to be framed base curves.

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KEYWORDS: spacelike curve, framed curve, framed immersion, pedal curve, contrapedal curve.

Introduction

In differential geometry, curves formed by curves are basic concepts with numerous applications in engineering and optics. Some of those are pedal curves and contrapedal curves. The pedal (contrapedal) curve is the locus of orthogonal projections from a given point to all of the tangent (normal) lines of a regular plane curve. There are several important uses for pedal curves in astronomy. Moreover, pedal curves are used in optics, notably in the construction of small camera lenses. Pedal and contrapedal curves can be obtained for singular curves as well as regular curves. However, when the curve has singularities, such as cusps, inflection points, etc., tangent lines are not well defined at these points [5]. So, the Frenet frame is not used to examine the local differential geometry of curves which have singularities. For this reason, S. Honda and M. Takahashi have defined the framed curves in Euclidean space [3]. Therefore, the singular curve in Euclidean 3-space is characterized using framed curves. Then, the existence conditions of framed curves for smooth curves have been given [2]. In addition to Frenet curves, Legendre curves are also being promoted in this way. Studies on this subject have been carried out for many different spaces. One of them has been done by P. Li, D. Pei [4] and they have shown the definition of spacelike framed curves in Minkowski 3-space. Moreover, they have defined the nullcone fronts of a spacelike framed curve. In addition, the pedal and contrapedal curves of framed immersions in Euclidean space are expressed in [5]. Authors provide the relationships between the evolutes and involutes of framed immersions, as well as pedal and contrapedal curves.

The main motivation of our work is that we encounter singular curves rather than regular curves in daily life. Therefore, solving problems on this subject is very important. Unlike previous studies, our aim is to investigate the pedal and contrapedal curves of spacelike framed immersions in Minkowski 3-space. For this, we obtain the Frenet type formulas for spacelike framed immersions by using the Frenet type formulas of framed curves. Then, we define pedal and contrapedal curves considering these formulas. Moreover, we obtain the necessary conditions for pedal and contrapedal curves of spacelike framed immersions to be framed base curves.

Preliminaries

Let $\mathbb{R}^3$ be the 3-dimensional real vector space. For $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, the *pseudo scalar product* of $\mathbf{a}$ and $\mathbf{b}$ is given by $\langle \mathbf{a}, \mathbf{b} \rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3$. Then, the vector space $\mathbb{R}^3$ associated with this pseudo scalar product is called a *3-dimensional pseudo Euclidean space, or Minkowski 3-space* and it is shown with $(\mathbb{R}^3, \langle , \rangle)$. Also, for the sake of the shortness, $\mathbb{R}^3_1$ is written instead of $(\mathbb{R}^3, \langle , \rangle)$.

The vector $\mathbf{a} \in \mathbb{R}^3_1$ is called *spacelike, lightlike* or *timelike* if $\langle \mathbf{a}, \mathbf{a} \rangle > 0$, $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ or $\langle \mathbf{a}, \mathbf{a} \rangle < 0$, respectively and the *norm of the vector* $\mathbf{a} \in \mathbb{R}^3_1$ is given by $\|\mathbf{a}\| = \sqrt{|\langle \mathbf{a}, \mathbf{a} \rangle|}$. Moreover, the signature of $\mathbf{a}$ is denoted by ε such that $\dfrac{\langle \mathbf{a}, \mathbf{a} \rangle}{\|\mathbf{a}\|^2} = \varepsilon$ [1] .

We have the *pseudo-spheres* with centered origin in $\mathbb{R}^3_1$ which are given by

$$S^2_1 = \{\mathbf{a} \in \mathbb{R}^3_1 | \langle \mathbf{a}, \mathbf{a} \rangle = 1\}, \tag{1}$$

which is called pseudo 2-sphere,

$$H^2_0 = \{\mathbf{a} \in \mathbb{R}^3_1 | \langle \mathbf{a}, \mathbf{a} \rangle = -1\}, \tag{2}$$

which is called hyperbolic 2-space [4].

Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^3_1$ be a regular curve, which is parametrized by an open interval I. For any $t \in I$, we say that the curve is *spacelike, timelike, lightlike* if $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}'(t) \rangle > 0$, $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}'(t) \rangle < 0, \langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}'(t) \rangle = 0$, respectively [4].

Accordingly, the following definition is given for a spacelike spatial curve with singular points.

Definition 1. *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a spacelike curve and a map (γ, ν_1, ν_2) be $(\gamma, \nu_1, \nu_2) : I \longrightarrow \mathbb{R}^3_1 \times \Delta$ where*

$$\Delta = \{(\nu_1, \nu_2) \in S^2_1 \times H^2_0 | \langle \nu_1(t), \nu_2(t) \rangle = 0\} \tag{3}$$

or

$$\Delta = \{(\nu_1, \nu_2) \in H^2_0 \times S^2_1 | \langle \nu_1(t), \nu_2(t) \rangle = 0\}. \tag{4}$$

If $\langle \gamma^{'}(t), \nu_1(t) \rangle = 0$ and $\langle \gamma^{'}(t), \nu_2(t) \rangle = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is called a spacelike framed curve [4].

Definition 2. *$\gamma(t)$ is called the original curve of the spacelike framed curve* [4].

Remark 2. *For a spacelike curve, we know that there are three types of frame fields. Only two are examined in this research.*

Assume that $\mu(t) = \nu_1(t) \wedge \nu_2(t)$ which is a spacelike vector field. In addition to, let $\alpha(t)$ be a smooth function such that $\gamma^{'}(t) = \alpha(t)\mu(t)$. Then the Frenet type formulas for a spacelike framed curve are given as following:

$$\begin{bmatrix} \nu_1^{'}(t) \\ \nu_2^{'}(t) \\ \mu^{'}(t) \end{bmatrix} = \begin{bmatrix} 0 & -\delta l_1(t) & -l_2(t) \\ -\delta l_1(t) & 0 & l_3(t) \\ \delta l_2(t) & \delta l_3(t) & 0 \end{bmatrix} \begin{bmatrix} \nu_1(t) \\ \nu_2(t) \\ \mu(t) \end{bmatrix}$$

where $\langle \nu_1(t), \nu_1(t) \rangle = sign(\nu_1(t)) = \delta$ and $\langle \nu_2(t), \nu_2(t) \rangle = sign(\nu_2(t)) = -\delta$. Also, the map $(\alpha(t), l_1(t), l_2(t), l_3(t))$ is the curvature of the spacelike framed curve which is given by

$$\begin{aligned} l_1(t) &= \langle \nu_1^{'}(t), \nu_2(t) \rangle, \\ l_2(t) &= \langle \mu^{'}(t), \nu_1(t) \rangle, \\ l_3(t) &= \langle \nu_2^{'}(t), \mu(t) \rangle, \\ \alpha(t) &= \langle \gamma^{'}(t), \mu(t) \rangle. \end{aligned}$$

Main results

Let $(\gamma, \nu_1, \nu_2) : I \longrightarrow \mathbb{R}^3_1 \times \Delta$ be a spacelike framed curve. Firstly, we obtain a spacelike framed immersion by rotating $\nu_1(t)$ and $\nu_2(t)$. For this, we use the plane spanned by $\nu_1(t)$ and $\nu_2(t)$, which is the normal plane of γ. According to Lorentzian rotation, we define

$$\begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix} = \frac{1}{\sqrt{|l_2^2(t) - l_3^2(t)|}} \begin{bmatrix} l_2(t) & l_3(t) \\ l_3(t) & l_2(t) \end{bmatrix} \begin{bmatrix} \nu_1(t) \\ \nu_2(t) \end{bmatrix}$$

where $l_2^2(t) - l_3^2(t) \neq 0$ for all $t \in I$. In that case, $n_1(t)$ and $\nu_1(t)$ have the same causal character. Therefore, $n_2(t)$ and $\nu_2(t)$ have the same causal character. Moreover, $(\gamma, \nu_1, \nu_2) : I \longrightarrow \mathbb{R}^3_1 \times \Delta$ is a spacelike framed immersion and $n_1(t) \times n_2(t) = \mu(t)$.

The Frenet type formulas of the spacelike framed immersion (γ, n_1, n_2) are given by

$$\begin{bmatrix} n_1'(t) \\ n_2'(t) \\ \mu'(t) \end{bmatrix} = \begin{bmatrix} 0 & L(t) & -M(t) \\ L(t) & 0 & 0 \\ \delta M(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1(t) \\ n_2(t) \\ \mu(t) \end{bmatrix}$$

The map $(L, M, 0, \alpha) : I \to \mathbb{R}^4$ is the curvature of (γ, n_1, n_2). The curvature functions are determined as

$$L(t) = \left[-\delta l_1(t) + \frac{l_2(t) l_3'(t) - l_2'(t) l_3(t)}{l_2^2(t) - l_3^2(t)} \right]$$

and

$$M(t) = \varepsilon \sqrt{|l_2^2(t) - l_3^2(t)|}.$$

While the osculating plane of γ at $\gamma(t)$ is the plane spanned by $\mu(t)$ and $n_1(t)$, the normal plane of γ at $\gamma(t)$ is the plane spanned by $n_1(t)$ and $n_2(t)$.

Assume that $(\gamma, n_1, n_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a spacelike framed immersion and $p \in \mathbb{R}^3_1$ is a fixed point. When we take the orthogonal projection point of p on each osculating plane of γ, we have a curve. Likewise, if we use the normal plane rather than the osculator plane, we have another curve. Now let us give the definitions of these special curves.

Definition 3. *Assume that $(\gamma, n_1, n_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a spacelike framed immersion and $p \in \mathbb{R}^3_1$ is a fixed point. Then, the pedal curve and the contrapedal curve are given as the followings:*
1) The pedal curve of the spacelike framed immersion (γ, n_1, n_2) associated with the point p is

$$Pe_{\gamma,p}(t) = p + \delta \langle \overrightarrow{\gamma(t)p}, n_2(t) \rangle n_2(t)$$

where the map $Pe_{\gamma,p}(t)$ is $Pe_{\gamma,p} : I \to \mathbb{R}^3_1$ and p is called pedal point.
2) The contrapedal curve of the spacelike framed immersion (γ, n_1, n_2) associated with the point p is

$$CPe_{\gamma,p}(t) = p - \langle \overrightarrow{\gamma(t)p}, \mu(t) \rangle \mu(t)$$

where the map $CPe_{\gamma,p}(t)$ is $CPe_{\gamma,p} : I \to \mathbb{R}^3_1$ and p is called contrapedal point.

Theorem 4. *Let (γ, n_1, n_2) be a spacelike framed immersion. Then, its pedal curve has a chausal character as the following:*
i) If $f_2^2(t) - f_1^2(t) > 0$ and n_1 is spacelike (timelike), then the pedal curve is spacelike (timelike),
ii) If $f_2^2(t) - f_1^2(t) < 0$ and n_1 is spacelike (timelike), then the pedal curve is timelike (spacelike),
where $p \in \mathbb{R}_1^3$ is a fixed point and $f, f_1, f_2 : I \to \mathbb{R}$ are smooth functions such that

$$\begin{aligned}\langle p-\gamma(t), n_1(t)\rangle &= f(t)f_1(t),\\ \langle p-\gamma(t), n_2(t)\rangle &= f(t)f_2(t),\\ f_2^2(t)-f_1^2(t) &\neq 0.\end{aligned}$$

Proof. When we take the derivative of the pedal curve $Pe_{\gamma,p}(t)$, we obtain as the following:

$$\begin{aligned}Pe^{'}_{\gamma,p}(t) =& -\delta\left\langle\gamma^{'}(t), n_2(t)\right\rangle n_2(t) + \delta\left\langle p-\gamma(t), n_2^{'}(t)\right\rangle n_2(t)\\ &+\delta\left\langle p-\gamma(t), n_2(t)\right\rangle n_2^{'}(t)\\ =& -\delta\left\langle\alpha(t)\mu(t), n_2(t)\right\rangle n_2(t) + \delta\left\langle p-\gamma(t), L(t)n_1(t)\right\rangle n_2(t)\\ &+\delta\left\langle p-\gamma(t), n_2(t)\right\rangle L(t)n_1(t)\\ =&\delta L(t)f(t)\left[f_2(t)n_1(t)+f_1(t)n_2(t)\right]\end{aligned}$$

Therefore, the causal character of the pedal curve is

$$\langle Pe^{'}_{\gamma,p}(t), Pe^{'}_{\gamma,p}(t)\rangle = \delta L^2(t)f^2(t)(f_2^2(t)-f_1^2(t)).$$

□

Theorem 5. *Let (γ, n_1, n_2) be a spacelike framed immersion, (L, M, O, α) be the curvature and $p \in \mathbb{R}_1^3$ be a fixed point. If there exist the smooth functions $f, f_1, f_2 : I \to \mathbb{R}$ such that*

$$\langle p-\gamma(t), n_1(t)\rangle = f(t)f_1(t),\ \langle p-\gamma(t), n_2(t)\rangle = f(t)f_2(t), f_2^2(t)-f_1^2(t)\neq 0,$$

then the pedal curve $Pe_{\gamma,p}$ is a framed base curve.

Proof. According to the proof of the previous theorem,

$$Pe^{'}_{\gamma,p}(t) = \delta L(t)f(t)\left[f_2(t)n_1(t)+f_1(t)n_2(t)\right].$$

When γ is spacelike, the pedal curve can be timelike or spacelike. Accordingly, $\widehat{\nu_1}$ and $\widehat{\nu_2}$ can also be timelike or spacelike. Considering these causal characters, we get

$$\widehat{\nu_1} = \mu(t)\ , \quad \widehat{\nu_2} = \frac{f_1(t)n_1(t)+f_2(t)n_2(t)}{\sqrt{|f_2^2(t)-f_1^2(t)|}}$$

or

$$\widehat{\nu_1} = \frac{f_1(t)n_1(t)+f_2(t)n_2(t)}{\sqrt{|f_2^2(t)-f_1^2(t)|}}\ , \quad \widehat{\nu_2} = \mu(t).$$

Then, $(Pe_{\gamma,p}, \widehat{\nu_1}, \widehat{\nu_2}) : I \to \mathbb{R}_1^3 \times \Delta$ is a framed curve. □

Theorem 6. *Let (γ, n_1, n_2) be a spacelike framed immersion. Then, its contrapedal curve has a chausal character as the following:*
i) If n_1 is spacelike (timelike), then the contrapedal curve is spacelike,
ii) If $g_2^2(t) - g_1^2(t) > 0$ ($g_2^2(t) - g_1^2(t) < 0$) and n_1 is timelike, then the contrapedal curve is spacelike (timelike),
where $p \in \mathbb{R}_1^3$ is a fixed point and $(g, g_1, g_2) : I \to \mathbb{R}$ are smooth functions such that

$$\alpha(t) - \delta M(t)\langle p - \gamma(t), n_1(t)\rangle = g(t)g_2(t),$$
$$-\delta M(t)\langle p - \gamma(t), \mu(t)\rangle = g(t)g_1(t),$$
$$g_2^2(t) + \delta g_1^2(t) \neq 0.$$

Proof. When we take the derivative of the contrapedal curve $CPe_{\gamma,p}(t)$, we obtain as the following:

$$\begin{aligned} CPe_{\gamma,p}^{'}(t) = & -\left\langle -\gamma^{'}(t), \mu(t)\right\rangle \mu(t)) - \left\langle p - \gamma(t), \mu^{'}(t)\right\rangle \mu(t) \\ & - \langle p - \gamma(t), \mu(t)\rangle \mu^{'}(t) \\ = & \langle \alpha(t)\mu(t), \mu(t)\rangle \mu(t) - \langle p - \gamma(t), \delta M(t) n_1(t)\rangle \mu(t) \\ & - \langle p - \gamma(t), \mu(t)\rangle \delta M(t) n_1(t) \\ = & \left[\alpha(t) - \delta\mu(t) \langle p - \gamma(t), n_1(t)\rangle\right] \mu(t) - \delta M(t) \langle p - \gamma(t), \mu(t)\rangle n_1(t) \end{aligned}$$

Therefore, the causal character of the contrapedal curve is

$$\langle CPe_{\gamma,p}^{'}(t), CPe_{\gamma,p}^{'}(t)\rangle = g^2(t)\left[g_2^2(t) + \delta g_1^2(t)\right].$$

□

Theorem 7. *Let (γ, n_1, n_2) be a spacelike framed immersion, (L, M, O, α) be the curvature and $p \in \mathbb{R}_1^3$ be a fixed point. If there exist the smooth functions $g, g_1, g_2 : I \to \mathbb{R}$ such that $\alpha(t) - \delta M(t)\langle p - \gamma(t), n_1(t)\rangle = g(t)g_2(t)$, $-\delta M(t)\langle p - \gamma(t), \mu(t)\rangle = g(t)g_1(t)$, $g_2^2(t)+\delta g_1^2(t) \neq 0$, then the contrapedal curve $CPe_{\gamma,p}$ is a framed base curve.*

Proof. According to the proof of the previous theorem,

$$CPe_{\gamma,p}^{'}(t) = g(t)\left[g_2(t)\mu(t) + g_1(t)n_1(t)\right].$$

When γ is spacelike, the pedal curve can be timelike or spacelike. Accordingly, $\widehat{\nu_1}$ and $\widehat{\nu_2}$ can also be timelike or spacelike. Considering these causal characters, we get

$$\overline{\nu_1} = n_2(t)\ , \quad \overline{\nu_2} = \frac{-\delta g_1(t)\mu(t) + g_2(t)n_1(t)}{\sqrt{|g_2^2(t) + \delta g_1^2(t)|}}$$

or

$$\overline{\nu_1} = \frac{-\delta g_1(t)\mu(t) + g_2(t)n_1(t)}{\sqrt{|g_2^2(t) + \delta g_1^2(t)|}}\ , \quad \overline{\nu_2} = n_2(t).$$

Then, $(CPe_{\gamma,p}, \overline{\nu_1}, \overline{\nu_2}) : I \to \mathbb{R}_1^3 \times \Delta$ is a framed curve. □

Conclusion

The pedal and contrapedal curves of spacelike framed immersions in Minkowski 3-space have been examined in this work. We investigate Frenet type formulas for spacelike framed immersions and using these formulas, we establish the pedal curves and the contrapedal curves. Additionally, we have the prerequisites that these curves to be framed base curves. Furthermore, we demonstrate that a pedal curve or a contrapedal curve of spacelike framed immersion does not always require a spacelike or a timelike.

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References

[1] G. Aydın Şekerci, *On evolutoids and pedaloids in Minkowski 3-space*, J. Geom. Phys. **168**, 2021; Article ID: 104313, https://doi.org/10.1016/j.geomphys.2021.104313.

[2] T. Fukunaga and M. Takahashi, *Existence conditions of framed curves for smooth curves*, J. Geom. **108**, 763–774, 2017; https://doi.org/10.1007/s00022-017-0371-5.

[3] S. Honda and M. Takahashi, *Framed curves in the Euclidean space*, Adv. Geom. **16 (3)**, 265–276, 2016; https://doi.org/10.1515/advgeom-2015-0035.

[4] P. Li and D. Pei, *Nullcone fronts of spacelike framed curves in Minkowski 3-space*, Mathematics **9**, 2939, 2021; https://doi.org/10.3390/math9222939.

[5] K. Yao, M. Li, E. Li and D. Pei, *Pedal and contrapedal curves of framed immersions in the Euclidean 3-space*, Mediterr. J. Math. **20**, 204, 2023; https://doi.org/10.1007/s00009-023-02408-z.

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Important points of the cluster analysis method in multivariate statistical analysis

Fusun Yalcin

Cluster analysis is a multivariate statistical method used in almost all branches of science. This article gives a brief description of the highlights of cluster analysis. An example of a cluster analysis dendogram is also given.

2020 MSC: 62H30, 62P30

KEYWORDS: Statistics, Multivariate atatistical methods, Cluster analysis

Introduction

Cluster analysis is a multivariate statistical technique widely used in social sciences, natural sciences, applied sciences and engineering. First, it is important to understand what clustering is. Clustering is defined as the grouping of a set of objects according to some internal characteristics. It is a grouping in which objects with the same characteristics are grouped together. The researcher can specify these characteristics. Although clustering analysis is thought of as just an algorithm, it is actually like a series of tasks under the algorithm. In other words, clustering analysis can be performed with different algorithms to determine what a cluster consists of in general and how to find these clusters in the most efficient way. The main purpose of cluster analysis is to discover the natural grouping of units from complex structured data sets and to classify the units into clusters according to whether they are homogeneous or not (*cf.* [1]). Cluster analysis was first introduced in anthropology in 1932 by Driver and Kroeber (*cf.* [3]) and was later used in psychology by Joseph Zubin in 1938 (*cf.* [11]) and Robert Tryon in 1939 (*cf.* [8]). Cattell used the trait theory classification in personality psychology in 1943 (*cf.* [9]).

Material and method

When applying cluster analysis, the first step should be to obtain observations for p variables from n units taken from the population for which there is no precise information about their natural groupings. In the second stage, the distances between the units or variables should be determined using an appropriate similarity measure that will help us to determine the similarities or differences between the units or variables (*cf.* [1, 7]). Depending on whether the variables are discrete or continuous, nominal, ordinal, interval or proportional, it is necessary to decide which distance or similarity measure to use.

With the appropriate clustering algorithm determined in the third step, the units or variables should be divided into an appropriate number of clusters according to similarity and difference matrices. While clustering methods divide units or variables into appropriate groups, they are divided into two basic groups according to the approaches they follow in determining the groups. These methods can be divided into two groups: Hierarchical Clustering Analysis Methods and Non-Hierarchical Clustering Analysis Methods. Finally, the clusters obtained are interpreted according to their structures and analysis is carried out (*cf.* [1, 7]). Hierarchical clustering methods are methods that aim to combine units with each other at certain levels, taking into account their similarities.

Non-hierarchical clustering is a method that aims to group units into clusters that are homogeneous within themselves and heterogeneous between themselves. In hierarchical clustering, both units and variables form clusters with different levels of similarity to each other, whereas in non-hierarchical methods, the aim is to group the units into appropriate clusters and to divide n units into k clusters (*cf.* [1, 7]). There are several types of distance formula used in cluster analysis, which are described in Everitt's book. If you are interested, you can use this book to study them in detail (*cf.* [4]).

Example 1. *In the study article we will give as an example, in the study of heavy metals in Akkaya Lake Reservoir Soil, a dendogram was created using Hierarchical Clustering Analysis, considering the chemical analysis results of samples taken from 31 different stations and the Euclidean distance coefficient (cf.* [10]*). Sites showing similar behaviour to each other formed three distinct groups. The first group: Site 5 is externally connected to group 10, 30, 24, 8, 28, 11, 25, 7, 21 and 20 (Figure* 1*) Second group: Locations 22 and 27. Third group: consists of 4 sub-groups within itself. First subgroup: 3, 14, 19 and 31; second subgroup: 1, 6, 4 and 13; third sub-group: 15, 17 and 29; fourth subgroup: 9, 16, 2, 12, 26, 23 and 18. Sites with similar characteristics may contain similar proportions of elements between them (Figure* 1*). Cluster analysis can help us to distinguish between elements according to their similarities, but it can also help us to get an idea of the distribution of the cluster formed by each element. More detailed examples can be found in the articles (cf.* [2, 4, 5, 6]*).*

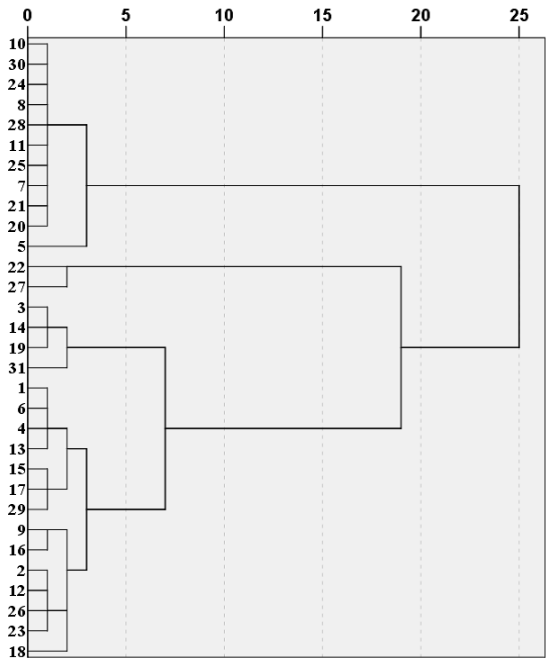

Figure 1: Dendogram showing the chemical analysis of the samples taken to study heavy metals in the Akkaya Reservoir Soil

Conclusion

In multivariate statistics, cluster analysis is an important tool for grouping variables or objects by the distances between them. In this study a brief information about cluster analysis is given. A sample suitable for the purpose of the study has been taken. The grouping of elements shown in the example is widely used in geological science.

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References

[1] M. R. Anderberg, *Cluster analysis for applications: Probability and mathematical statistics: A series of monographs and textbooks* (Volume 19), Academic press, 2014.

[2] R. K. Blashfieldy and M. S.Aldenderfer, *The literature on cluster analysis*, Multivariate behavioral research **13 (3)**, 271–295, 1978.

[3] H. E. Driver and A. L. Kroeber, *Quantitative expression of cultural relationships*, Berkeley, University of California Press **31 (4)**, 1932.

[4] B. S. Everitt, S. Landau, M. Leese and D. Stahl, *Cluster analysis*, John Wiley and Sons, 2011.

[5] A. Jaeger and D. Banks, *Cluster analysis: A modern statistical review*, Wiley Interdisciplinary Reviews, Computational Statistics **15 (3)**, e1597, 2023.

[6] H. F. Köhn and L. J. Hubert, *Hierarchical cluster analysis*, Wiley StatsRef, Statistics reference online **1 (13)**, 2014.

[7] S. Sharma, *Applied multivariate techniques*, John Wiley and Sons, Inc., 1995.

[8] R. C. Tryon, *Cluster analysis: Correlation profile and orthometric (factor) analysis for the isolation of unities in mind and personality*, Edwards brother, Incorporated Ann Arbor, 1939.

[9] Wikipedia, *Cluster analysis*, Available at https://en.wikipedia.org/wiki/Cluster_analysis#cite_note-1, (Accession date: 12 Jan 2024).

[10] F. Yalcin, *Data analysis of heavy metals in Akkaya lake reservoir soils using multivariate statistical analysis*, Turkish Journal of Agriculture-Food Science and Technology **9 (1)**, 249–257, 2021.

[11] J. Zubin, *A technique for measuring like-mindedness*, The Journal of Abnormal and Social Psychology **33 (4)**, 508, 1938.

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Numbers of the digital age: New media and statistical approaches

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The spread and development of digital technologies has shown an unusually rapid and continuous development. New media, which is at the center of this development, have become an integral part of daily life. We discussed statistics' critical role in understanding new media phenomena in the digital age. In this article, we examine how new media becomes a data source in various areas, from social media platforms to interactive websites and mobile applications to online marketing and advertising, and how these data are interpreted through statistical analysis. In particular, it discusses how statistics are used in collecting, processing, and interpreting new media data and the importance of data analysis. The interaction between new media and statistics is important not only for data scientists and marketers but also for other segments of society. The study also serves as a guide for anyone who wants to understand the developments in new media and the digital facts behind these developments in a data-driven world.

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KEYWORDS: New media, Social media, Statistics, Big data, Q-Learning

Introduction

New media emerges as one of the most important concepts of the 21st century. In 2022, 96% of young people in the EU used the internet daily; this rate was 84% in the entire population [5]. New media is defined as interactive, versatile, virtuality and networked mass media provided by digital technology such as the Internet [8, 20]. The concept covers all content accessed via computers, smartphones and tablets [4]. While new media emphasizes the dynamic and user interaction of communication, traditional media such as newspapers, radio and television emphasize one-way communication and fixed content distribution. But as new technology is improved and widely accepted, what is considered "new" continues to change [4]. While VHS/VCR players, DVDs, CDs and mp3s were the most popular means of watching movies and listening to music in the past, today they are replaced by streaming services such as Netflix, Amazon Prime Video, Disney +, Max, Paramount +, and Spotify [15]. The rise of the world wide web, and especially the invention of powerful search engines, and social media platforms has led to explosive growth in social network size and connectivity; whereas in human history, social networks were small and local, organizations changed slowly, and power was concentrated in a small subset of the population [6].

New media are generally conceptualized as forms of media that rely heavily on digital technologies to communicate with audiences. Flew's research has focused on the transformative impact of digital age technologies on traditional media forms and the emergence of new media characterized by their digital nature [7]. The changing media landscape encompasses a variety of platforms, including blogs, augmented and virtual reality, computer games, mobile apps, video sharing programs, podcasts, streaming services, social media, chat platforms, vlogs, emails, forums, and websites [19].

Although the new media concept seems new, its emergence in academic literature dates back to the 1980s [17]. This seminal work offers a comprehensive analysis of new media, highlighting its dynamic nature and the transformation of communication methods over time. This evolution in definition reflects a shift from early understandings of new media as technology-based, interactive networks to the contemporary view. This narrative can be framed within the broader context of the development of digital communications, acknowledging the seminal works of the 1980s that form the basis for today's nuanced definition [26]. This approach places the concept within a historical and evolving scientific discourse and shows how technological advances shape and redefine our understanding of new media. Therefore, these studies are important for early recognition of the impact of digital technology on communication practices.

The relationship between new media and statistics is crucial to understanding modern communication dynamics. As new media platforms increase, so does the production of big data sets that require advanced statistical approaches to analyze user behavior and content trends. This relationship underscores the increasing importance of statistical methods in unraveling complex patterns resulting from digital media use. This data includes insights into user behavior, popular topics, market trends, etc., and can be used to identify patterns. Statistical methods and data analysis techniques are one of the tools that will help identify specific patterns and trends in these data sets. It is quite important to develop and use mathematical functions and models to model complex relationships and patterns in data. For example, user interactions and web traffic data can be used to understand the factors that influence the popularity of a particular website. By tracking traffic sources, measuring user engagement and monitoring conversions, the website can be optimized into a powerful tool to maximize leads and increase revenue [9]. Web analytics relates to the act of tracking, analyzing and generating reports on the use of a website, such as web pages, images or videos [21]. Recently, various machine learning approaches have been frequently used to overcome regression issues. Regression formulates relationships mathematically so that other values may be predicted using the equation [21].

This study focuses on the statistical analysis of large data sets obtained from new media sources, especially data collected through social media, web analytics, and other digital platforms. It aims to understand and apply these analyses effectively. Additionally, the study will detail how new media data can be evaluated, particularly in areas such as pattern recognition, trend analysis, and building predictive models. It will also explore how these techniques can be integrated with the discovery and analysis of mathematical functions.

Preliminaries

In the study, general definitions will be made of the statistical analysis of big data sets obtained from new media sources, especially data collected through social media, web analytics and other digital platforms. Strong statistical methods, popularly termed data mining or machine learning, enable the computer to detect complex patterns in data, which makes the application of statistical methods a fundamental step in generating knowledge with big data [13]. The sources and structure of new media data are presented in Figure 1. By analyzing data from social media, web analytics, digital advertising, multimedia content, mobile application, e-commerce, and content sharing in-depth information can be provided on user behavior, market trends, social trends, and many more. Complex new media data is often processed, analyzed and interpreted using big data analysis techniques and various statistical methods.

Statistical analysis and Machine Learning Methods, Cluster Analysis, Natural Language Processing (NLP), text mining and language modeling techniques are applied along with analysis such as Sentiment Analysis, Trend Analysis, Network Analysis, Topic Modeling. Results are presented with visualizations and interpretations. Each step can be adapted according to the purpose and specific needs of the analysis. Since new media data analysis is an ever-changing field, it is necessary to keep up with current tools and methods. This study will focus on forecasting methods and mathematical modeling often used to extract meaningful information from big data sets and predict future trends or behaviors. Each of these analyses is the subject of extensive research, and it is not possible to examine them all in a single study. In this study, where we focus on a specific place by getting a little more detailed than general, we will touch on frequently used forecasting methods to extract meaningful information from large data sets, predict future trends or behaviors, and focus on the Q-Learning algorithm.

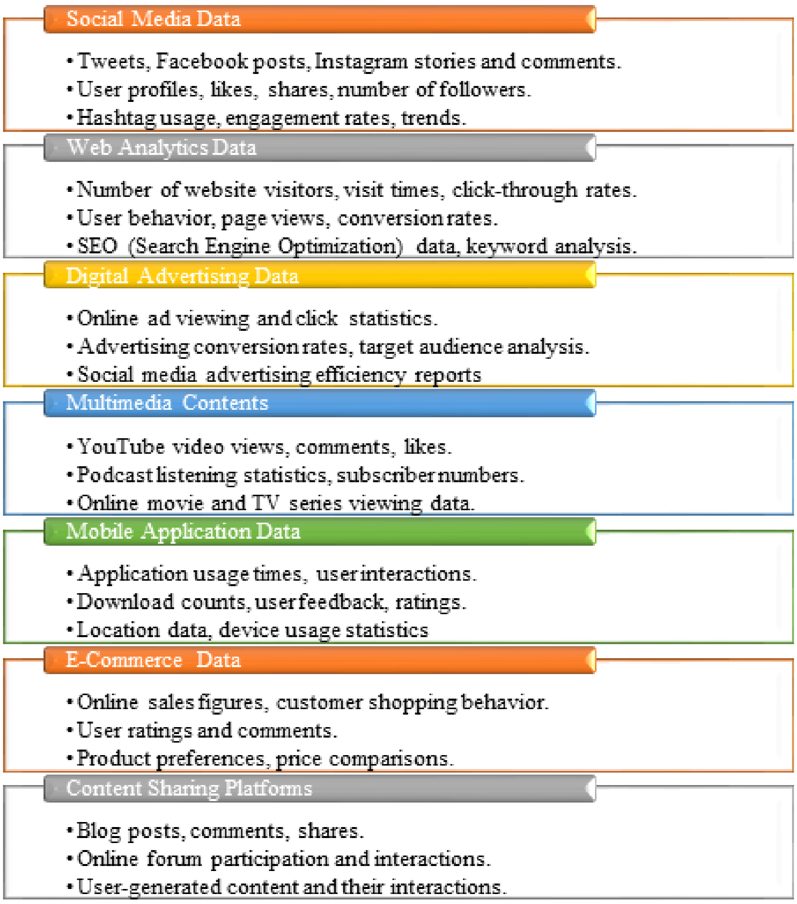

Figure 1: A selection of new media data sources

Some standard forecasting methods used to extract meaningful information that predict future trends or behavior are given in Figure 2. Integrating Machine Learning and Deep Learning with statistical analysis methods has been a significant area of interest in the research literature over the last decade. This integration is closely related to rapid developments in data science and the analysis of big data sets. Data mining has become an essential part of analysis as the speed and capacity of data generation increases [10].

Figure 2: A selection of forecasting methods for new media data

Social media data contains a wide range of information that can be used as a predictive tool [18]. Predictive analytics uses various statistical or data mining methods to make predictions about future outcomes and performance [12, 11]. Predictive power refers to the ability to accurately predict new observations. Predictive power is significant, and statistical models and algorithms for best prediction continue to be developed.

Main result

Statistics is a powerful tool that delves into the depths of new media data, illuminating various areas, from user behaviors to market trends. Algorithms also play an important role in the analysis of logical and simple mathematical operations. In this section we will focus on the applications and possible outcomes of the Q-learning approach to new media, which has been discussed with its pros and cons and is thought to take developments in artificial intelligence a step further. Q-learning [24] is one of the most popular reinforcement learning algorithms of recent times, although its theoretical development dates to the 1980s. Q-learning is an approach to Machine Learning that allows a model to learn and improve over time by taking the right actions and making the right decisions [3]. Q-Learning is an algorithm in the reinforcement learning category and is generally used to learn optimal actions to achieve a specific goal. Due to its potential applications in various fields, including new media, Q-learning has received considerable attention in recent years [1]. It was first introduced by Bellman (1957) [2] and later adapted to complex environments with unknown transition dynamics by approximate dynamic programming with function approximators [3]. Secondly, Watkins and Dayan (1992) introduced the original Q-learning algorithm [25]. This is an incremental (stochastic approximation) method for estimating the Q-function in a Markov decision process [16]. Previously limited to the computer science and control theory literature, these methods have provided a bridge from prediction methods in statistics to the application of Q-learning [14].

Mathematics: the Q-learning algorithm

To solve sequential decision problems, predictions for the optimal value of each action can be learned; this value is defined as the expected sum of future rewards when taking that action and following the optimal policy thereafter [23].

The Q-function uses the Bellman equation and takes two inputs: state (s) and action (a).

$$\underbrace{Q^{\pi}\left(s_t, a_t\right)}_{\text{Q-value for the state given a particular}} = E\left[\underbrace{R_{t+1}+\gamma R_{t+2}+\gamma^2 R_{t+3}+}_{\text{Expected discounted cumulative reward}} \quad \cdots \quad \underbrace{\mid s_t, a_t}_{\text{Given the state and action}}\right]. \tag{1}$$

The discount factor γ, which ranges from 0 to 1, balances the significance of immediate and future rewards. An optimal policy is obtained by selecting the action with the highest value in each state. A policy π defines the conditional probability distribution of choosing different actions depending on each state s. The reward sequence distribution can be determined once a stationary policy has been chosen. The evaluation of policy π can be achieved through an action-value function. This function is defined under π as the expected cumulative discounted reward when taking an action from state s and following the π policy. The optimal value for Q is $Q_*(s,a) = (\max_{\pi})Q_{\pi}\left(s,a\right)$. Banach's fixed-point theorem can obtain the uniqueness and existence of the fixed-point solution of Bellman optimality equations.

$$Q_*(s,a) = R\left(s,a\right) + \gamma \int_{s'} T\left(s' \mid s,a\right) \max_{a'} Q_*(s',a'). \tag{2}$$

When interacting with the environment at time t, the reinforcement learning agent observes information about the current state s and chooses an action a based on a policy. It then receives a reward r from the environment based on the action taken and transitions to a new state s'. Q-learning is a model-independent approach that updates the prediction of Q-values based on experience instances at each time step using the following equation.

$$Q(s,a) \leftarrow Q(s,a) + a\left(r + \gamma \max_{a'} Q_*(s',a') - Q(s,a)\right). \tag{3}$$

Here, α represents the learning rate, and $Q(s,a)$ denotes the current estimation. Q-learning, a form of temporal difference learning, can be used to learn estimates of optimal action values [22]. The standard Q-learning update, after taking the A_t action in the S_t state and observing the R_{t+1} reward and the S_{t+1} outcome state immediately afterwards, is done for the parameters as follows:

$$\theta_{t+1} = \theta_t + \alpha\left(Y_t^Q - Q\left(S_t, A_t; \theta_t\right)\right) \nabla_{\theta_t} Q\left(S_t, A_t; \theta_t\right) \tag{4}$$

The scalar step size is represented by α and the target Y_t^Q is defined as:

$$Y_t^Q = R_{t+1} + \gamma \max_a Q\left(S_{t+1}, A_t; \theta_t\right). \tag{5}$$

This update is similar to stochastic gradient descent which updates the current value $Q\left(S_t, A_t; \theta_t\right)$ towards the target value Y_t^Q.

The algorithm selects an action for the current state, receives a reward, and updates the Q-value at each iteration. This process continues until the state space is adequately explored or another stopping criterion is reached.

There are studies showing that Q-learning has the potential to improve decision-making processes and improve user experience in new media [27]. Possible areas of use of the Q-learning approach in new media studies in the future can be listed as follows. Q-learning can be used to learn what content to share to maximize user interaction on a social media platform. The algorithm can be used for User Interaction Optimization to recommend the best content by learning the effects of different content types on user engagement. By analyzing user behavior and preferences, Q-learning can help new media platforms make data-driven decisions that improve user engagement and satisfaction. The algorithm can be effective in making personalized content recommendations based on the user's previous interactions and preferences to recommend content that users will be interested in. The algorithm can learn which strategies drive the highest conversion rate or engagement. Q-learning can be used to predict users' future behavior. This could be, for example, predicting the likelihood that a user will express a positive or negative sentiment about a particular topic or product.

Advantages: Because it is a model-free approach, Q-learning can be effective in complex state spaces without needing model knowledge.

Challenges: Estimating and storing Q-values can be difficult in large state and/or action spaces.

Conclusion

This study highlights the importance of predictive models designed to predict new/future observations or scenarios, as well as methods and algorithms to evaluate the predictive power of the model, in addition to transforming new media data into high-quality data suitable for statistical analysis. New media, mainly social media analysis, is about more than understanding metrics or consumer sentiment - it is about navigating the intricacies of human behavior online. The nature of fields means that trends and user behavior can change instantly. This rapid volatility is reminiscent of the stock market; both require agile strategies to capitalize on emerging patterns. But while stock trading is mainly about analyzing numbers, social media is about dissecting emotions, interests, and interactions. It's like deciphering the digital DNA of society at large. In today's rapidly changing technological world, science will be relatively slower to keep up with this change. As in previous studies, it does not seem possible to produce results that do not change for many years in many areas. We believe that branches of science, especially those dealing with technology-based and highly interactive tools such as new media, will continue with studies that will trigger or develop the next study. Although the research findings in these fields rapidly become outdated, they will trigger other studies and pave the way for new studies and developments. Therefore, the speed of science production will also increase. This rapid change in current issues over time will affect many fields of science and accelerate multidisciplinary studies. As in this study, big data content produced as a result of the transformation of traditional into new media-based communication tools, has been integrated with statistical and mathematical methods.

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References

[1] H. Alavizadeh, H. Alavizadeh and J. Jang-Jaccard, *Deep Q-learning based reinforcement learning approach for network intrusion detection*, Computers **11 (3)**, 2022; Article ID: 41.

[2] R. Bellman, *Dynamic programming*, Princeton Univ. Press, Princeton, NJ, 1957.

[3] J. Clifton and E. Laber, *Q-learning: Theory and applications,* Annual Review of Statistics and Its Application **7**, 279–301, 2020.

[4] J. Cote, *What is new media?*, 2022; https://www.snhu.edu/about-us/newsroom/liberal-arts/what-is-new-media.

[5] Eurostat, *Being young in Europe today-digital world*, 2023; https://ec.europa.eu/eurostat/statistics-explained/index.php?title=Being_young_in_Europe_today_-_digital_world.

[6] J. C. Flack and R. M. D'Souza, *The digital age and the future of social network science and engineering*, Proceedings of the IEEE, 1**02 (12),** 1873–1877, 2014.

[7] T. Flew, *New media: An introduction*, Oxford University Press, 2008.

[8] N. Gane and D. Beer, *New media: The key concepts*, Berg. 2008.

[9] T. Gordon, *The importance of analyzing website traffic*, 2023; https://www.linkedin.com/pulse/importance-analyzing-website-traffic-tracewell-trace-gordon.

[10] M. K. Gupta and P. Chandra, *A comprehensive survey of data mining,* International Journal of Information Technology **12 (4)**, 1243–1257, 2020.

[11] C. Halton, *Predictive analytics: Definition, model types, and uses*, 2023; https://www.investopedia.com/terms/p/predictive-analytics.asp.

[12] E. Kalampokis, E. Tambouris and K. Tarabanis, *Understanding the predictive power of social media*, Internet Research **23 (5)**, 544–559, 2013.

[13] T. Lengauer, *Statistical data analysis in the era of big data*, Chemie Ingenieur Technik **92 (7)**, 831–841, 2020; https://doi.org/10.1002/cite.202000024.

[14] S. A. Murphy, *A generalization error for Q-learning*, Journal of Machine Learning Research **6**, 1073–1097, 2005.

[15] P. Nickinson, *The 10 most popular streaming services, ranked by subscriber count*, 2023; https://www.digitaltrends.com/home-theater/most-popular-streaming-services-by-subscribers.

[16] M. L. Puterman, *Markov decision processes: Discrete stochastic dynamic programming*, John Wiley & Sons, 2014.

[17] S. Rafaeli, *From new media to communication*, Sage annual review of communication research, Advancing communication science, **16 (1)**, 110–134, 1988.

[18] H. Schoen, D. Gayo-Avello, P. Takis Metaxas, E. Mustafaraj, M. Strohmaier and P. Gloor, *The power of prediction with social media*, Internet Research **23 (5)**, 528–543, 2013.

[19] J. Spacey, *50 Examples of new media*, 2021; https://simplicable.com/en/new-media-examples

[20] M. Shao, *Measurement and trend analysis of new media coverage topics based on comment big data mining*, Mathematical Problems in Engineering, 1–8, 2022; https://doi.org/10.1155/2022/1404468.

[21] D. Sikka and C. N. S. Vinoth Kumar, *Website traffic time series forecasting using regression machine learning,* 12th IEEE International Conference on Communication Systems and Network Technologies, 246–250, 2023.

[22] R. S. Sutton and A. G. Barto, *The reinforcement learning problem,* In: Reinforcement learning: An introduction, Cambridge, MA, MIT Press, 51–85, 1998.

[23] H. Van Hasselt, A. Guez and D. Silver, *Deep reinforcement learning with double q-learning*, In: Proceedings of the AAAI conference on artificial intelligence, **30 (1)**, 2016.

[24] C. J. C. H. Watkins, *Learning from delayed rewards*, PhD Thesis, University of Cambridge, England, 1989.

[25] C. J. Watkins and P. Dayan, *Q-learning,* Machine Learning **8**, 279–292, 1992.

[26] F. Williams, R. E. Rice and E. M. Rogers, *Research methods and the new media,* Simon and Schuster, 1988.

[27] Y. Zhu, *Flattening of new media design based on deep reinforcement learning*, Mobile Information Systems 2022.

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Some results for poly-Cauchy numbers and polynomials

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This research introduces some properties of a new class for the generalized poly-Cauchy numbers and polynomials. We present various recurrence relations, explicit formulae, and generating functions for these numbers and polynomials. Additionally, we discuss the corresponding generalized m-poly-Bernoulli numbers and polynomials related to the generalized m-poly-Cauchy numbers and polynomials.

2020 MSC: 05A15, 11B73, 11B68

KEYWORDS: Cauchy numbers and polynomials, Poly-Cauchy numbers, Poly-Bernoulli numbers, Recurrence relation, Stirling numbers

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References

[1] A. Z. Broder, *The r-Stirling numbers*, Discrete Math. **49**, 241–259, 1984.

[2] L. Carlitz, *Weighted Stirling numbers of the first and second kind. I*, Fibonacci Quart. **18**, 147–162, 1980.

[3] L. Carlitz, *Weighted Stirling numbers of the first and second kind. II*, Fibonacci Quart. **18**, 242–257, 1980.

[4] M. Cenkci and P. T. Young, *Generalizations of poly-Bernoulli and poly-Cauchy numbers*, Eur. J. Math. **1**, 799–828, 2015.

[5] L. Comtet, *Advanced combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.

[6] G. Guettai, D. Laissaoui and M. Rahmani, *Generalized Laguerre transforms of sequences*, Creat. Math. Inform. **32**, 1–6, 2023.

[7] H. Kaneko and T. Komatsu, *Cauchy-Carlitz numbers*, J. Number Theory **163**, 238–254, 2016.

[8] T. Komatsu, *Hypergeometric Cauchy numbers*, Int. J. Number Theory **9**, 545–560, 2013.

[9] T. Komatsu, *Poly-Cauchy numbers*, Kyushu J. Math. **67**, 143–153, 2013.

[10] T. Komatsu, *Poly-Cauchy numbers with a q parameter*, Ramanujan J. **31**, 353–371, 2013.

[11] T. Komatsu, V. Laohakosol and K. Liptai, *A generalization of poly-Cauchy numbers and their properties*, Abstr. Appl. Anal. **8**, 2013; Article ID: 179841.

[12] T. Komatsu, I. Mező and L. Szalay, *Incomplete Cauchy numbers*, Acta Math. Hungar. **149**, 306–323, 2016.

[13] T. Komatsu and L. Szalay, *Shifted poly-Cauchy numbers*, Lith. Math. J. **54**, 166–181, 2014.

[14] D. Merlini, R. Sprugnoli and M. C. Verri, *The Cauchy numbers*, Discrete Math. **306**, 1906–1920, 2006.

[15] M. Rahmani, *Generalized Stirling transform*, Miskolc Math. Notes **15**, 677–690, 2014.

[16] M. Rahmani, *On p-Cauchy numbers*, Filomat **30** , 2731–2742, 2016.

[17] M. Rahmani, *On p-Bernoulli numbers and polynomials*, J. Number Theory **157**, 350–366, 2015.

[18] M. Sebaoui, G. Guettai, D. Laissaoui, M. Rahmani, *Degenerate Stirling numbers and a family of Bell polynomials*, Mathematica **64 (87)**, 277–290, 2022.

[19] F.-Z. Zhao, *Some results for generalized Cauchy numbers*, Util. Math. **82**, 269–284, 2010.

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Associated polynomials arising from Laguerre transform

Ghania Guettai *[1], *Laissaoui Difallah* [2] *and Mourad Rahmani* [3]

In this work, we define and study a new class of polynomials known as associated polynomials, which exhibit properties closely resembling those of associated Laguerre polynomials. Initially, we extend the Laguerre transform to two variables. As a practical application, we define and investigate a new class of polynomials called two-variable associated Fibonacci polynomials.

2020 MSC: 11B39, 05A19, 33C45

KEYWORDS: Laguerre transform, Associated Laguerre polynomials, Fibonacci polynomials, Explicit formulas, Generating functions

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This paper is dedicated to Professor Taekyun KIM on the occasion of his 60th anniversary.

References

[1] D. Dattoli, *Generalized polynomials, operational identities and their applications*, J. Comput. Appl. Math. **118**, 111–123, 2000.

[2] D. Dattoli and A. Torre, *Operatorial methods and two variable Laguerre polynomials*, Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. **132**, 3–9, 1998.

[3] G. Guettai, D. Laissaoui and M. Rahmani, *Generalized laguerre transforms of sequences*, Creat. Math. Inform. **32**, 1–6, 2023.

[4] N. Khan, T. Usman and J. Choi, *A new class of generalized polynomials associated with Laguerre and Bernoulli polynomials*, Turk. J. Math. **43**, 486–497, 2019.

[5] T. Koshy, *Fibonacci and Lucas numbers with applications Volume II*, Pure Appl. Math., Wiley Ser. Texts Monogr. Tracts. Hoboken, NJ: John Wiley & Sons, 2019.

[6] N. N. Lebedev, *Special functions and their applications*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1965.

[7] J. Riordan, *An introduction to combinatorial analysis*, Princeton University Press, Princeton, New Jersey, 1980.

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Berezin radius inequalities with application of the contraction operator

Hamdullah Başaran

This study utilizes the contraction operator K to prove a few inequalities. Furthermore, a few inequality utilizing the arithmetic mean-geometric mean inequality (also known as AM-GM inequality) is given. The relationship between the spectral radius and the Berezin number inequalities of with using the contraction operator K is presented.

2020 MSC: 47A30, 47A63

KEYWORDS: Berezin radius, Spectral radius, Contraction operator, AM-GM inequality

Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle ., . \rangle)$. In the case when $\dim(\mathcal{H}) = n$, we identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices having entries the complex field. Throughout the work, we examine on reproducing kernel Hilbert space (RKHS), which are complete inner product spaces made up of complex-valued functions defined on a non-empty set Ω with bounded point evaluation. If $\langle Mx, x \rangle > 0$ for all $x \in \mathcal{H}$, then an operator $M \in \mathcal{B}(\mathcal{H})$ is called positive, and then we write $M > 0$. Also, the absolute value of positive operator is denoted by $|M| = (M^*M)^{\frac{1}{2}}$. Let Ω be a subset of a topological space X such that boundary $\partial\Omega$ is nonempty. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an infinite-dimensional Hilbert space of functions defined on Ω. We say that $\mathcal{H}$ is a reproducing kernel Hilbert space (briefly, RKHS) if the following two conditions are satisfied:

(i) for any $\lambda \in \Omega$, the functionals $f \to f(\lambda)$ are continuous on $\mathcal{H}$;

(ii) for any $\lambda \in \Omega$, there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda)$.

According to the classical Riesz representation theorem, the assumption (i) implies that for any $\lambda \in \Omega$ there exists $k_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle, \ f \in \mathcal{H}.$$

The function k_λ is called the reproducing kernel of $\mathcal{H}$ at point λ. Note that by (ii), we surely have $k_\lambda \neq 0$ and we denote $\widehat{k}_\lambda$ as the normalized reproducing kernel, that is $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$.

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions.

Definition 1. *Let $\mathcal{H}$ be a RKHS on a set Ω and let M be a bounded linear operator on $\mathcal{H}$.*

(i) *For* $\lambda \in \Omega$, *the Berezin transform of* M *at* λ *(or Berezin symbol of* M*) is*

$$\widetilde{M}(\lambda) := \left\langle M\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}}.$$

(ii) *The Berezin range of* M *(or Berezin set of* M*) is*

$$\mathrm{Ber}(M) := \mathrm{Range}(\widetilde{M}) = \left\{ \widetilde{M}(\lambda) : \lambda \in \Omega \right\}.$$

(iii) *The Berezin radius of* M *(or Berezin number of* M*) is*

$$\mathrm{ber}(M) := \sup_{\lambda \in \Omega} \left| \widetilde{M}(\lambda) \right|$$

(for more facts abouts reproducing kernel Hilbert spaces and Berezin symbol, see, Aronzajn [3] *and Berezin* [11]*).*

The Berezin transform $\widetilde{M}$ is a bounded real-analytic function on Ω for each bounded operator M on $\mathcal{H}$. The Berezin transform $\widetilde{M}$ frequently reflects the characteristics of the operator M. Since F. Berezin first proposed the Berezin transform in [11], it has become an essential tool in operator theory due to the fact that the Berezin transforms of many significant operators include information about their fundamental characteristics. It is said that Karaev initially explicitly introduced the Berezin set and Berezin number in [18], also denoted as $\mathrm{Ber}(M)$ and $\mathrm{ber}(M)$, respectively.

In a RKHS, the Berezin range of an operator M is a subset of the numerical range of T. Hence $\mathrm{ber}(M) \leq w(M)$. An operator's numeric range has a number of intriguing characteristics. For instance, it is common knowledge that an operator's numerical range's closure contains the operator's spectrum. We refer to [1, 2, 12, 20, 21] for the fundamental attributes of the numerical radius.

For example, is it true, or under which additional conditions the following are true:

(i) $\mathrm{ber}(M) \geq \frac{1}{2} \|M\|$;

(ii)

$$\mathrm{ber}(M^n) \leq \mathrm{ber}(M)^n \tag{1}$$

for any integer $n \geq 1$; more generally, if M is not nilpotent, then

$$C_1 \mathrm{ber}(M)^n \leq \mathrm{ber}(M^n) \leq C_2 \mathrm{ber}(M)^n$$

for some constants $C_1, C_2 > 0$;

(iii) $\mathrm{ber}(MN) \leq \mathrm{ber}(M)\,\mathrm{ber}(N)$, where $M, N \in \mathcal{B}(\mathcal{H})$.

If $M = cI$ with $c \neq 0$, then obviously $\mathrm{ber}(M) = |c| > \dfrac{|c|}{2} = \dfrac{\|M\|}{2}$. However, it is known that in general the above inequality (i) is not satisfied (see Karaev [19]).

It is well-known that

$$\frac{1}{2} \|M\| \leq w(M) \leq \|M\| \tag{2}$$

and

$$\mathrm{ber}(M) \leq w(M) \leq \|M\| \tag{3}$$

for any $M \in \mathcal{B}(\mathcal{H})$. Additionally, Karaev introduced additional numerical properties of operators on the RKHS in [18], including Berezin range and Berezin radius. See [9, 13, 14, 15, 17, 22] for the fundamental characteristics and information on these novel concepts.

In 2018, Bakherad, has showed Berezin radius inequalities of block matrix of the form $\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ (see, [4, 5, 7]). In 2022, Huban et al. [16] have proven the following results:

$$\mathrm{ber}\left(M\right) \leq \frac{1}{2}\left\|\left|M\right| + \left|M^*\right|\right\|_{\mathrm{ber}} \tag{4}$$

and Başaran et al. [10] have showed the following inequality:

$$\mathrm{ber}^2\left(M\right) \leq \frac{1}{2}\left\|\left|M\right|^2 + \left|M^*\right|^2\right\|_{\mathrm{ber}}.$$

As one can see in [1, 6, 8], where operator norm and numerical radius inequalities were researched and implemented, operator matrices and their properties and inequalities have attracted a lot of attention in the literature. Inequalities for the Berezin number of operator matrices have just lately been introduced in [4, 5].

The direct sum of two copies of $\mathcal{H}$ is defined by $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$. If $P, R, S, T \in \mathcal{B}\left(\mathcal{H}\right)$, then the operator matrix $M = \begin{bmatrix} P & R \\ S & T \end{bmatrix}$ can be considered as on operator in $\mathcal{B}\left(\mathcal{H} \oplus \mathcal{H}\right)$, which denoted by $My = \begin{bmatrix} P & R \\ S & T \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Px_1 + Rx_2 \\ Sx_1 + Tx_2 \end{bmatrix}$ for every vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{H} \oplus \mathcal{H}$.

This study utilizes the contraction operator K to prove a few inequalities. Furthermore, a few inequality utilizing the arithmetic mean-geometric mean inequality (also known as AM-GM inequality) is given. The relationship between the spectral radius and the Berezin number inequalities of with using the contraction operator K is presented.

Main results

Corollary 2. *Let $M \in \mathcal{B}\left(\mathcal{H}\left(\Upsilon_1\right) \oplus \mathcal{H}\left(\Upsilon_2\right)\right)$ and let f and g be non-negative functions on $[0, \infty)$ which are continouns with $f\left(t\right) g\left(t\right) = t$ for all $t \in [0, \infty)$. Then there is a contraction $K \in \mathcal{B}\left(\mathcal{H}\left(\Upsilon_1\right) \oplus \mathcal{H}\left(\Upsilon_2\right)\right)$ such that*

$$\mathrm{ber}\left(M\right) = \mathrm{ber}\left(M^*\right) \leq \frac{1}{2}\left\|g\left(\left|M^*\right|\right) K^* K g\left(\left|M^*\right|\right) + f^2\left(\left|M\right|\right)\right\|_{\mathrm{ber}}.$$

In particular, for any $0 \leq \upsilon \leq 1$

$$\mathrm{ber}\left(M\right) = \mathrm{ber}\left(M^*\right) \leq \frac{1}{2}\left\|\left|M^*\right|^{1-\upsilon} K^* K \left|M^*\right|^{1-\upsilon} + \left|M\right|^{2\upsilon}\right\|_{\mathrm{ber}}.$$

Corollary 3. *Let $T \in \mathcal{B}\left(\mathcal{H}\left(\Upsilon_1\right) \oplus \mathcal{H}\left(\Upsilon_2\right)\right)$ and let f and g be non-negative functions on $[0, \infty)$ which are continouns with $f\left(t\right) g\left(t\right) = t$ for all $t \in [0, \infty)$. Then there is a contraction $K \in \mathcal{B}\left(\mathcal{H}\left(\Upsilon_1\right) \oplus \mathcal{H}\left(\Upsilon_2\right)\right)$ such that*

$$\mathrm{ber}^2\left(T\right) \leq r\left(f^2\left(\left|T\right|\right) K^* g^2\left(\left|T^*\right|\right) K\right),$$

where $r\left(.\right)$ is the spectral radius, then

(i) $\mathrm{ber}^2\left(T\right) = \mathrm{ber}^2\left(g\left(\left|T^*\right|\right) K f\left(\left|T\right|\right)\right)$,

(ii) $\operatorname{ber}^2\left(g\left(|T^*|\right)Kf\left(|T|\right)\right) \leq r\left(f^2\left(|T|\right)K^*g^2\left(|T^*|\right)K\right)$.

In particular, for any $0 \leq \upsilon \leq 1$

(i) $\operatorname{ber}^2\left(T\right) = \operatorname{ber}^2\left(|T^*|^{1-\upsilon}K|T|^{\upsilon}\right)$,

(ii) $\operatorname{ber}^2\left(|T^*|^{1-\upsilon}K|T|^{\upsilon}\right) \leq r\left(|T|^{2\upsilon}K^*|T^*|^{2(1-\upsilon)}K\right)$.

Conclusion

In this study, inequalities were obtained using the contraction operator K. Also, inequalities are shown using the arithmetic mean-geometric mean inequality (also known as the AM-GM inequality). The relationship between spectral radius and Berezin number inequalities was found using the contraction operator K. In the future, it is aimed to prove two-operator versions of these inequalities.

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References

[1] A. Abu-Omar, F. Kittaneh, *Numerical radius inequalities for $n \times n$ operator matrices*, Linear Algebra Appl. **468**, 18–26, 2015.

[2] E. Alizade and A. Farokhina, *Some refinements of numerical radius inequalities for Hilbert space operators*, Jordan J. Math. Stat. **15 (1)**, 55–63, 2022.

[3] N. Aronzajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68**, 337–404, 1950.

[4] M. Bakherad, *Some Berezin number inequalities for operator matrices*, Czechoslovak Math. J. **68 (4)**, 997–1009, 2018.

[5] M. Bakherad and M. T. Garayev, *Berezin number inequalities for operators*, Concr. Oper. **6 (1)**, 33–43, 2019.

[6] M. Bakherad and K. Shebrawi, *Upper bounds for numerical radius inequalities involving off-diagonal operator matrices*, Ann. Funct. Anal. **9 (3)**, 297–309, 2018.

[7] M. Bakherad, M. Hajmohamadi, R. Lashkaripour and S. Sahoo, *Some extensions of Berezin number inequalities on operators*, Rocky Mountain J. Math. **51 (6)**, 1941–1951, 2021.

[8] W. Bani-Domi and F. Kittaneh, *Norm and numerical radius inequalities for Hilbert space operators*, Linear Multilinear Algebra **69 (5)**, 934–945, 2021.

[9] H. Başaran and V. Gürdal, *Berezin radius and Cauchy-Schwarz inequality*, Montes Taurus J. Pure Appl. Math. **5 (3)**, 16–22, 2023.

[10] H. Başaran, M. B. Huban and M. Gürdal, *Inequalities related to Berezin norm and Berezin number of operators*, Bull. of Math. Anal. and App. **14 (2)**, 1–11, 2022.

[11] F. A. Berezin, *Covariant and contravariant symbols for operators*, Math. USSR-Izvestiya. **6**, 1117–1151, 1972.

[12] P. Bhunia and K. Paul, *Refinements of norm and numerical radius inequalities,* Rocky Mountain J. Math. **51 (6)**, 1956–1965, 2021.

[13] M. T. Garayev, H. Guedri, M. Gürdal and G. M. Alsahli, *On some problems for operators on the reproducing kernel Hilbert space*, Linear Multilinear Algebra **69 (11)**, 2059–2077, 2021.

[14] V. Gürdal and H. Başaran, *On Berezin radius inequalities via Cauchy-Schwarz type inequalities*, Malaya J. Mat. **11 (2)**, 127–141, 2023.

[15] M. B. Huban, H. Başaran and M. Gürdal, *New upper bounds related to the Berezin number inequalities*, J. Inequal. Spec. Funct. **12 (3)**, 1–12, 2021.

[16] M. B. Huban, H. Başaran and M. Gürdal, *Some new inequalities via Berezin numbers*, Turk. J. Math. Comput. Sci. **14 (1)**, 129–137, 2022.

[17] M. B. Huban, H. Başaran and M. Gürdal, *Berezin number inequalities via convex functions*, Filomat **36 (7)**, 2333–2344, 2022.

[18] M. T. Karaev, *Berezin symbol and invertibility of operators on the functional Hilbert spaces*, J.Funct. Anal. **238**, 181–192, 2006.

[19] M. T. Karaev, *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory. **7**, 983–1018, 2013.

[20] F. Kittaneh and H. R. Moradi, *Cauchy-Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl. **22 (3)**, 1117–1125, 2020.

[21] F. Kittaneh, H. R. Moradi and M. Sababheh, *Mean inequalities for the numerical radius,* Numer. Funct. Anal. Optim. **44 (14)**, 1523–1537, 2023.

[22] S. S. Sahoo, N. Das and D. Mishra, *Berezin number and numerical radius inequalities for operators on Hilbert spaces*, Advances in Oper. Theory. **5**, 714–727, 2020.

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A study on dimorphic properties of Bernoulli random variable

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In this talk, we study a dimorphic property associated with two different sums of identically independent Bernoulli random variables having two different families of probability mass functions. There are various ways of studying special numbers and polynomials, to mention a few, generating functions, combinatorial methods, probability theory, p-adic analysis, umbral calculus, differential equations, special functions and analytic number theory. In recent years, we have had lively interests in the study of various degenerate versions of special numbers and polynomials with those diverse tools. As a fruit of such explorations, we came up with, for example, the degenerate Stirling numbers which are degenerate versions of the ordinary Stirling numbers and appear in many different contexts.

2020 MSC: 11B73, 60G50

KEYWORDS: Bernoulli random variable, Dimorphic properties, Degenerate Stirling numbers of the second kind, Stirling numbers of the first kind

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References

[1] T. Kim, D. S. Kim, H. Lee and S.-H. Park, *Dimorphic Properties of Bernoulli Random Variable*, Filomat **36 (5)**, 1711–1717, 2022.

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New explicit formulas of the degenerate two variables Fubini polynomials

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In this paper, we explore some explicit formulas for the higher-order degenerate two variables Fubini polynomials and numbers in terms of the degenerate Stirling numbers of the second kind, the degenerate r-Stirling numbers and polynomials of the second kind, and the generalized degenerate falling factorials, which is different from the previous works. In addition, we introduce a new type degenerate poly-Fubini polynomials of two variables and investigate some interesting identities for them.

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KEYWORDS: Fubini polynomials, Stirling numbers

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A note on generalized Bernoulli polynomials of the second kind

Han Young Kim *[1], *Lee Chae Jang* [2] *and Jongkyum Kwon* [3]

Recently, generalized Bernoulli polynomials and Bernoulli polynomials of the second kind are introduced by Kim-Kim. In this paper, we consider generalized Bernoulli polynomials of the second kind. The aim of this paper is to derive, for those polynomials, some properties, recurrence relations, and explicit expressions. Generalized Bernoulli polynomials of the second kind in connection with the Stirling number of the first kind, Stirling number of the second kind, the Daehee numbers and the Harmonic numbers.

2020 MSC: 11B68, 11B75, 11B73

KEYWORDS: Generalized degenerate type 2 Euler polynomials, The generalized degenerate type 2 Euler-Genocchi polynomials of order α, The degenerate Stirling numbers of the second kind

Introduction

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \text{(see [1,3])} \tag{1}$$

When $x = 0, B_n = B_n(0)$ are called the Bernoulli number.

For $r \in \mathbb{N}$, the Bernoulli polynomials of order r are defined by the generating function to be

$$\begin{aligned}\left(\frac{t}{e^t-1}\right)^r e^{xt} &= \underbrace{\left(\frac{t}{e^t-1}\right)\times\cdots\times\left(\frac{t}{e^t-1}\right)}_{r-times} e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \quad \text{(see [3,6,7])}\end{aligned} \tag{2}$$

In particular, if $r = 1$, $B_n(x) = B_n(1)$ are the ordinary Bernoulli polynomials. When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r.

As is well known, the Bernoulli polynomials of the second kind (or the Cauchy polynomials) are given by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see [2,3,4,7])} \tag{3}$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind.

From (2) and (3), we easily see that $b_n = B_n^{(n)}(1)$ $(n \geq 1)$. For a nonnegative integer n the stirling number of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad \text{(see [1,3,10,11])}. \tag{4}$$

Where $(x)_0 = x, (x)_n = x(x+1)\dots(x-n+1)$, $(n \geq 1)$. By the direct computation of (4) we derive the following

$$\frac{1}{n!}\left(\log(1+t)\right)^n = \sum_{k=n}^{\infty} S_1(k,n)\frac{t^k}{k!}, \quad \text{(see [4,5,16,17])}. \tag{5}$$

For a given nonnegative integer n, the Stirling number of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad \text{(see [3,10,13])}. \tag{6}$$

By (6), we obtain

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n) = \frac{t^l}{l!}, \quad \text{(see [12,13])}. \tag{7}$$

As is known, the Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \quad \text{(see [2,4,8,9,10])}. \tag{8}$$

When $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.
It is well known that the harmonic numbers are defined by

$$H_0 = 0,\ H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad (n \geq 1), \quad \text{(see [5])}. \tag{9}$$

From (9), we can derive the generating function of harmonic numbers given by

$$-\frac{1}{1-t}\log(1-t) = \sum_{n=1}^{\infty} H_n t^n \quad \text{(see [13,14])}. \tag{10}$$

The aim of this paper is to study several relations among those four kind of numbers. We develop methods for generalized Bernoulli polynomials of the second kind, we represent the generalized Bernoulli numbers in terms of the Stirling number of the first kind, the Stirling number of the second kind, the Daehee numbers and the Harmonic numbers. We deduce a recurrence relation for generalized Bernoulli polynomials.

The generalized Bernoulli polynomials of the second kind

For $r \in \mathbb{N}$, the generalized Bernoulli polynomials of the second kind are defined by the generating function to be

$$\frac{t^r}{\log(1+t)}(1+t)^x = \sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!}. \tag{11}$$

When $x = 0$, $c_n^{(r)} = c_n^{(r)}(0)$ are called the generalized Bernoulli numbers of the second kind. Note that $c_0^{(r)}(x) = c_1^{(r)}(x) = c_2^{(r)}(x) = \cdots = c_{r-2}^{(r)}(x) = 0$. When $r = 1$, we get that $c_n^{(1)}(x) = c_n(x)$ for $n = 0, 1, 2 \ldots$.

By (11), we observe that

$$\begin{aligned}
\sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^x = t^{r-1} \times \frac{t}{\log(1+t)}(1+t)^x \\
&= t^{r-1}\sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} c_n(x)\frac{t^{n+r-1}}{n!} \\
&= \sum_{n=r-1}^{\infty} c_{n-r+1}(x)\frac{n!}{(n-r+1)!}\frac{t^n}{n!} \\
&= \sum_{n=r-1}^{\infty} (r-1)!\binom{n}{r-1} c_{n-r+1}(x)\frac{t^n}{n!}. \qquad (12)
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (12), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{N}$, we have*

$$c_n^{(r)} = \begin{cases} 0, & \text{if } n < r-1 \\ (r-1)!\binom{n}{r-1} c_{n-r+1}(x), & \text{if } n \geq r-1 \end{cases}.$$

From (11) and (12), we observe that

$$\begin{aligned}
\sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^x \\
&= (r-1)!\frac{(\log(1+t))^{r-1}}{(r-1)!}\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x \\
&= (r-1)!\sum_{m=r-1}^{\infty} S_1(m, r-1)\frac{t^m}{m!}\sum_{l=0}^{\infty} B_n^{(r)}(x)\frac{t^l}{l!} \qquad (13) \\
&= (r-1)\sum_{n=r-1}^{\infty}\left(\sum_{l=r-1}^{n}\binom{n}{l} B_l^{(r)}(x) S_1(n-l, r-1)\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus, comparing the coefficients on both sides of (13) we obtain the following theorem.

Theorem 2. *For $n \geq 0$,*

$$c_n^{(r)}(x) = (r-1)!\sum_{l=r-1}^{n}\binom{n}{l} B_l^{(r)}(x) S_1(n-l, r-1).$$

Now, from (11), we get that

$$\begin{aligned}
\sum_{n=r-1} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{t^r}{\log(1+t)}(1+t)^x \\
&= \left(\sum_{m=r-1}^{\infty} c_m^{(r)}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\binom{x}{l} t^l\right) \\
&= \sum_{n=r-1}^{\infty}\left(\sum_{l=r-1}^{n}\binom{n}{l}\binom{x}{l} c_{n-l}^{(r)}\right)\frac{t^n}{n!}. \qquad (14)
\end{aligned}$$

Therefore, by comparing coefficient on both sides of (14), we obtain the following theorem.

Theorem 3. *For $n \geq 0$ and $r \in \mathbb{N}$ we have*

$$c_n^{(r)}(x) = \sum_{l=r-1}^{n} \binom{n}{l} \binom{x}{l} c_{n-l}^{(r)}.$$

For $x, y \in \mathbb{R}$, we observe that

$$\begin{aligned} \sum_{n=r-1}^{\infty} c_n^{(r)}(x+y) \frac{t^n}{n!} &= \frac{t^r}{\log(1+t)} (1+t)^{x+y} \\ &= \frac{t^r}{\log(1+t)} (1+t)^x (1+t)^y \\ &= \left(\sum_{m=r-1}^{\infty} c_m^{(r)}(x) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \binom{y}{l} t^l \right) \\ &= \sum_{n=r-1}^{\infty} \left(\sum_{l=r-1}^{n} \binom{n}{l} \binom{y}{l} c_{n-l}^{(r)}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{15}$$

Therefore, by comparing coefficient on both sides of (15), we obtain the following theorem.

Theorem 4. *For $x, y \in \mathbb{R}$ and $n \geq 0$ we have*

$$c_n^{(r)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} \binom{y}{l} c_{n-l}^{r}(x).$$

By (5) and (11), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} c_n^{(r)}(x) \frac{t^n}{n!} &= \frac{t^r}{\log(1+t)} (1+t)^x \\ &= \frac{t^r}{\log(1+t)} \times \frac{t}{\log(1+t)} \frac{\log(1+t)}{t} (1+t)^x \\ &= \left(\sum_{m=r-1}^{\infty} c_m^{(r)} \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} b_l \frac{t^l}{l} \right) \left(\sum_{i=0}^{\infty} D_i(x) \frac{t^i}{i!} \right) \\ &= \sum_{k=r-1}^{\infty} \left(\sum_{m=r-1}^{k} \binom{k}{m} c_m^{(r)} b_{k-m} \right) \frac{t^k}{k!} \left(\sum_{i=0}^{\infty} D_i(x) \frac{t^i}{i!} \right) \\ &= \sum_{n=r-1}^{\infty} \left(\sum_{k=r-1}^{n} \left(\sum_{m}^{k} c_m^{(r)} b_{k-m} D_{n-k}(x) \right) \right) \frac{t^n}{n!}. \end{aligned} \tag{16}$$

Therefore, by comparing the coefficients on both sides of(16) we get the following theorem.

Theorem 5. *For $n \geq 0$, we have*

$$c_n^{(r)}(x) = \left(\sum_{k=r-1}^{n} \left(\sum_{m}^{k} c_m^{(r)} b_{k-m} D_{n-k}(x) \right) \right).$$

We observe that

$$
\begin{aligned}
t^r(1+t)^x &= \sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} \times \log(1+t) \\
&= \left(\sum_{m=r-1}^{\infty} c_m^{(r)}\frac{t^m}{m!}\right)\left(\sum_{l=1}^{\infty}\frac{(-1)^l}{l}t^l\right) \\
&= \sum_{n=r}^{\infty}\left(\sum_{m=r}^{n}\frac{c_m^{(r)}(x)}{m!}\frac{(-1)^{n-m}}{(n-m)}\right)t^n
\end{aligned} \tag{17}
$$

and

$$
(1+t)^x t^r = \sum_{n=0}^{\infty}\binom{x}{n}t^{n+r} = \sum_{n=r}^{\infty}\binom{x}{n-r}t^n. \tag{18}
$$

By (17) and (18), we obtain the following theorem.

Theorem 6. *For $n \geq r$, we have*

$$
\sum_{m=r}^{n}\frac{b_m^{(r)}(x)}{m!}\frac{(-1)^{n-m}}{(n-m)} = \binom{x}{n-r}.
$$

Replacing t by e^t-1 in (11), we get

$$
\begin{aligned}
\frac{(e^t-1)}{t}\times e^{tx} &= \sum_{m=r-1}^{\infty} c_m^{(r)}(x)\frac{(e^t-1)^m}{m!} \\
&= \sum_{m=r-1}^{\infty} c_m^{(r)}(x)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\
&= \sum_{n=r-1}^{\infty}\left(\sum_{m-r-1}^{n}\binom{n}{m}c_m^{(r)}(x)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{19}
$$

On the left hand sides of (19), we have

$$
\begin{aligned}
\frac{(e^t-1)^r}{t}e^{tx} &= \frac{r!}{t}\frac{(e^t-1)^r}{r!}e^{tx} \\
&= \frac{r!}{t}\left(\sum_{m=r}^{\infty} S_2(m,r)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{x^l t^l}{l!}\right) \\
&= \frac{r!}{t}\sum_{n=r}^{\infty}\left(\sum_{m=r}^{n}\binom{n}{m}S_2(m,r)x^{n-m}\right)\frac{t^n}{n!} \\
&= r!\sum_{n=r}^{\infty}\left(\sum_{m=r}^{n}\binom{n}{m}S_2(m,r)x^{n-m}\right)\frac{t^{n-1}}{n!} \\
&= r!\sum_{n=r-1}^{\infty}\left(\sum_{m=r-1}^{n+1}\binom{n+1}{m}S_2(m,r)\frac{x^{n-m+1}}{(n+1)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{20}
$$

By (19) and (20), we obtain the following theorem.

Theorem 7. *For $n \geq r-1$, we have*

$$
\sum_{m-r-1}^{n}\binom{n}{m}c_m^{(r)}(x)S_2(n,m) = \sum_{m=r-1}^{n+1}\binom{n+1}{m}S_2(m,r)\frac{x^{n-m+1}}{(n+1)}.
$$

From (10) and (11), we can derive the following equation

$$\begin{aligned}
\sum_{n=r-1}^{\infty} c_n^{(r)}(x)\frac{t^n}{n!} &= \frac{\log(1+t)}{1+t}\frac{t^r}{(\log(1+t))^2}(1+t)^{x+1} \qquad (21)\\
&= \left(\sum_{l=1}^{\infty}(-1)^{l-1}H_l t^l\right)\left(\frac{t}{\log(1+t)}\frac{t^r}{\log(1+t))}(1+t)^{x+1}\right)\\
&= \left(\sum_{l=1}^{\infty}(-1)^{l-1}H_l t^l\right)\left(\sum_{l_1=0}^{\infty} b_{l_1}\frac{t^{l_1}}{l_1!}\right)\left(\sum_{l_2=0}^{\infty} c_{l_2}^{(r-1)}(x+1)\frac{t^{l_2}}{l_2!}\right)\\
&= \left(\sum_{l=1}^{\infty}(-1)^{l-1}H_l t^l\right)\sum_{k=0}^{\infty}\left(\sum_{l_1=0}^{k} b_{l_1} c_{k-l_1}^{(r-1)}(x+1)\binom{k}{l_1}\right)\frac{t^k}{k!}\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\sum_{l_1=0}^{k}\binom{k}{l_1} b_{l_1} c_{k-l_1}^{(r-1)}(x+1)(-1)^{n-k}n!\frac{H_{n-k}}{k!}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 8. *For $n \geq 1$ we have,*

$$c_n^{(r)}(x) = \sum_{k=0}^{n-1}\sum_{l_1=0}^{k}\binom{k}{l_1} b_{l_1} c_{k-l_1}^{(r-1)}(x+1)(-1)^{n-k}n!\frac{H_{n-k}}{k!},$$

where H_n are the Harmonic numbers.

Now we observe that

$$\begin{aligned}
\frac{t^r}{\log(1+t)}(1+t)^x &= \frac{\log(1+t)}{1+t}\cdot\frac{t^r}{(\log(1+t))^2}(1+t)^{x+1}\\
&= \frac{\log(1+t)}{1+t}\frac{\log(1+t)}{1+t}\cdot\frac{t^r}{(\log(1+t))^3}(1+t)^{x+2}\\
&= \frac{\log(1+t)}{1+t}\frac{\log(1+t)}{1+t}\frac{\log(1+t)}{1+t}\cdot\frac{t^r}{(\log(1+t))^4}(1+t)^{x+3}\\
&= \cdots
\end{aligned}$$

continuing this process, we get

$$\begin{aligned}
&\underbrace{\frac{\log(1+t)}{1+t}\times\cdots\times\frac{\log(1+t)}{1+t}}_{r-1\ times}\cdot\frac{t^r}{(\log(1+t))^r}(1+t)^{x+r-1}\\
&=\left(\sum_{l_1=1}^{\infty}(-1)^{l_1-1}H_{l_1}t^{l_1}\ \sum_{l_2=1}^{\infty}(-1)^{l_2-1}H_{l_2}t^{l_2}\times\cdots\times\sum_{l_{r-1}=1}^{\infty}(-1)^{l_{r-1}-1}H_{l_{r-1}}t^{l_{r-1}}\right)\\
&\quad\times\left(\sum_{k=r-1}^{\infty}c_n^{(r)}(x+r-1)\frac{t^n}{n!}\right)\\
&=\sum_{m=r-1}^{\infty}\left(\sum_{l_1+\cdots+l_{r-1}=m}(-1)^{m+r-1}H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\right)t^m\left(\sum_{k=r-1}^{\infty}c_n^{(r)}(x+r-1)\frac{t^n}{n!}\right)\\
&=\sum_{n=r-1}^{\infty}\left(\sum_{m=r-1}^{n}\sum_{l_1+\cdots+l_{r-1}=m}(-1)^{m+r-1}H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\frac{c_{n-m}^{(r)}(x+r-1)n!m!}{m!(n-m!)}\right)\frac{t^n}{n!}\\
&=\sum_{n=r-1}^{\infty}\left(\sum_{m=r-1}^{n}\sum_{l_1+\cdots+l_{r-1}=m}(-1)^{m+r-1}H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\binom{n}{m}c_{n-m}^{(r)}(x+r-1)m!\right)\frac{t^n}{n!}
\end{aligned}\tag{22}$$

Therefore, we obtain the following theorem.

Theorem 9. *For $n \geq 1$ we have,*

$$c_n^{(r)}(x)=\begin{cases}0, & \textit{if } n<r-1,\\ \sum_{m=r-1}^{n}\sum_{l_1+\cdots+l_{r-1}=m}(-1)^{m-r+1} & \\ \times H_{l_1}H_{l_2}\cdots H_{l_{r-1}}\binom{n}{m}m!c_{n-m}^{(r)}(x+r-1), & \textit{if } n\geq r-1.\end{cases}$$

Acknowledgments

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References

[1] S. Araci, *A new class of Bernoulli polynomials attached to polyexponential functions and related identities*, Stud. Contemp. Math. **31 (2)**, 195–204, 2021.

[2] B. El-Desouky and A. Mustafa, *New results on higher-order Daehee and Bernoulli numbers and polynomials* Adv. Differ. Equ. **2016** 2016; Atricle ID: 32.

[3] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math **15**, 51–88, 1979.

[4] D. Gun and Y. Simsek, *Combinatorial sums involving Stirling, Fubini, Bernoulli numbers and approximate values of Catalan numbers*, Adv. Stud. Contemp. Math. **30 (4)** 503–513, 2020.

[5] W. A. Khan, K. S. Nisar, U. Duran, M. Acikgoz and S. Araci, *Multifarious implicit summation formulae of Hermite-based poly-Daehee polynomials*, Appl. Math. Inf. Sci. **12 (2)**, 305–310, 2018.

[6] D. S. Kim and T. Kim, *Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation*, Bull. Korean Math. Soc. **52 (6)**, 2000–2010, 2015.

[7] T. Kim and D. S. Kim, *Daehee Numbers and polynomials*, Appl. Math. Sci. **7 (120)**, 5969–5976 2013; http://dx.doi.org/10.12988/ams.2013.39535

[8] T. Kim and D. S. Kim *Some relations of two type 2 polynomials and discrete harmonic numbers and polynomials*, Symmetry **12 (6)**, 2020; Article ID: 905, https://doi.org/10.3390/sym12060905.

[9] T. Kim and D. S. Kim *Some identities on degenerate hyperharmonic numbers*, Georgian Math. J. **30 (2)**, 255–262, 2023; https://doi.org/10.1515/gmj-2022-2203.

[10] T. K. Kim and D. S. Kim *Some identities involving degenerate Stirling numbers associated with several degenerate polynomials and numbers*, Russ. J. Math. Phys. **30**, 62–75, 2023.

[11] T. Kim, D. S. Kim and H. K. Kim *Some identities related to degenerate Stirling numbers of the second kind*, **55 (1)**, 812–821, 2022; https://doi.org/10.1515/dema-2022-0170.

[12] T. Kim, D. S. Kim, H. Y. Kim and J. Kwon, *A new type degenerate Daehee numbers and polynomials*, 2020; ArXiv:2004.08743.

[13] D. S. Kim, T. Kim, J. Kwon and H. Lee, *A note on λ-Bernoulli numbers of the second kind*, Adv. Stud. Contemp. Math. **30 (2)** 187–195, 2020.

[14] D. S. Kim, T. Kim, S.-H. Lee and J.-J. Seo, *Higher-order Daehee numbers and polynomials*, Int. J. Math. Anal. **8**, 273–283, 2014.

[15] T. Kim, D. S. Kim, H. Y. Kim and J. Kwon, *Degenerate Stirling polynomials of the second kind and some applications*, Symmetry **11 (8)**, 2019; Article ID: 1046.

[16] Y. Ma, D. S. Kim, Y. Lee, S. Park and T. Kim, *A study on multi-stirling numbers of the first kind*, Fractals **30 (10)**, 2022; Article ID: 2240258.

[17] Y. Simsek, *Identities and relations related to combinatorial numbers and polynomials*, Proc. Jangjeon Math. Soc. **20 (1)**, 127–135, 2017.

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Results originating from the application of Euler's formula to the generating functions for q-combinatorial Simsek polynomials

Irem Kucukoglu

The main aim of this presentation is to derive new finite sums through the application of the Euler's formula to the generating functions for q-combinatorial Simsek polynomials.

2020 MSC: 05A15, 05A19, 05A30, 11B65, 11B83

KEYWORDS: Combinatorial Simsek numbers and polynomials, q-analogues, q-analysis, Generating functions, Computation formulas, Finite sums, Trigonometric functions, Euler's formula

Introduction

In this presentation, the author aims to introduce new finite sums involving higher-power of q-binomial coefficients together with sine and cosine functions. Before achieving this goal, we begin to recall the definition of q-binomial coefficients:

$$\left[\begin{array}{c} k \\ j \end{array}\right]_q := \frac{[k]_q!}{[j]_q!\,[k-j]_q!}; \qquad (j=0,1,\ldots,k) \tag{1}$$

where $[k]_q!$ denotes the q-factorial:

$$[k]_q! := \begin{cases} 1 & \text{if } \ k=0, \\ [k]_q\,[k-1]_q \ldots [2]_q\,[1]_q & \text{if } \ k\in\mathbb{N} \end{cases} \tag{2}$$

in which

$$[k]_q := \frac{1-q^k}{1-q} = 1+q+q^2+\cdots+q^{k-1} \tag{3}$$

for $q\in\mathbb{C}\setminus\{1\}$ and a nonnegative integer k (*cf.* [4]; and also see [2, 3]).

In [7], the author have recently handled finite sums involving higher-power of q-binomial coefficients and introduced the q-combinatorial Simsek numbers $y_{6,q}(n,k;\lambda,r)$ and the q-combinatorial Simsek polynomials $y_{6,q}(x;n,k;\lambda,r)$ of the sixth kind respectively by the following finite sums:

$$y_{6,q}(n,k;\lambda,r) = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \left[\begin{array}{c} k \\ j \end{array}\right]_q^r [j]_q^n\,\lambda^j \tag{4}$$

and

$$y_{6,q}(x;n,k;\lambda,r) = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \left[\begin{array}{c} k \\ j \end{array}\right]_q^r [x+j]_q^n\,\lambda^j, \tag{5}$$

where $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) and n, k, r are nonnegative integers (*cf.* [7]).

Observe that

$$y_{6,q}(n,k;\lambda,r) = y_{6,q}(0;n,k;\lambda,r).$$

As the author also stated in [7], by setting $r = 1$ in (4) and (5), we have

$$y_{6,q}(n,k;\lambda,1) = y_{1,q}(n,k;\lambda) \tag{6}$$

and

$$y_{6,q}(x;n,k;\lambda,1) = y_{1,q}(x;n,k;\lambda) \tag{7}$$

where $y_{1,q}(n,k;\lambda)$ and $y_{1,q}(x;n,k;\lambda)$ denotes, in turn, the q-combinatorial Simsek numbers and polynomials of the first kind which have been recently introduced by the author [6] by the following finite sums:

$$y_{1,q}(n,k;\lambda) = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^n \lambda^j \tag{8}$$

and

$$y_{1,q}(x;n,k;\lambda) = \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [x+j]_q^n \lambda^j. \tag{9}$$

As stated in [7], when q goes to 1, the equation (4) and (5) implies

$$\lim_{q\to 1} y_{6,q}(n,k;\lambda,r) = y_6(n,k;\lambda,r) = \frac{1}{k!}\sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right)^r j^n \lambda^j$$

and

$$\lim_{q\to 1} y_{6,q}(x;n,k;\lambda,r) = P(x;n,k;\lambda,r) = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) x^{n-j} y_6(j,k;\lambda,r).$$

where $y_6(n,k;\lambda,r)$ and $P(x;n,k;\lambda,r)$ denotes, in turn, the combinatorial Simsek numbers and polynomials of the sixth kind which have been recently introduced and investigated by Simsek in [10] and [12].

Besides, as stated in [7], when q goes to 1 and $r = 1$, the equation (4) yields

$$\lim_{q\to 1} y_{6,q}(n,k;\lambda,1) = y_1(n,k;\lambda)$$

where $y_1(n,k;\lambda)$ denote the combinatorial Simsek numbers and polynomials of the first kind recently introduced and investigated by Simsek in [11].

In addition to the above special case, as stated in [7], when q goes to 1 and $\lambda = 1$, the equation (4) gives

$$\lim_{q\to 1} y_{6,q}(n,k;1,r) = \frac{1}{k!}\sum_{j=0}^{k} \binom{k}{j}^r j^n$$

which is of a wide variety relationships with some special finite sums. To see the mentioned relationships, the reader may refer to [12] and the references cited therein.

Additionally, as stated in [7], when $r = 1$ and $\lambda = -1$, the equation (4) also yields

$$y_{6,q}(n,k;-1,1) = q^{\binom{k}{2}} S_{2,q}(n,k) \tag{10}$$

where $S_{2,q}(n,k)$ denotes the q-Stirling numbers of the second kind defined by Carlitz in [1], and the numbers $S_{2,q}(n,k)$ are expressed with the help of the following formulae:

$$S_{2,q}(n,k)=\frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q [k-j]_q^n \tag{11}$$

and

$$[x]_q^n=\sum_{k=0}^{n}q^{\binom{k}{2}}\begin{bmatrix}x\\k\end{bmatrix}_q [k]_q! S_{2,q}(n,k) \tag{12}$$

(*cf.* [1, 5, 9]).

For detailed analysis, special cases and features of the q-combinatorial Simsek numbers and polynomials, the reader may refer to the author's recent works [6], [7], [8].

Main results

In this study, we first introduce an exponential generating functions for the q-combinatorial Simsek polynomials $y_{6,q}(x;n,k;\lambda,r)$ of the sixth kind:

$$\mathcal{G}_{y_{6,q}}(t,x;k;\lambda,r):=\sum_{n=0}^{\infty}y_{6,q}(x;n,k;\lambda,r)\frac{t^n}{n!}. \tag{13}$$

The combination of (13) with (5) gives

$$\mathcal{G}_{y_{6,q}}(t,x;k;\lambda,r)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\left[\begin{array}{c}k\\j\end{array}\right]_q^r \lambda^j\sum_{n=0}^{\infty}[x+j]_q^n\frac{t^n}{n!}. \tag{14}$$

The infinite series on the right-hand side of (14) is the Taylor series expansion of the function $\exp\left([x+j]_q t\right)$. Therefore, from the above equation we deduce the following theorem:

Theorem 1. *Let $k\in\mathbb{N}_0$. Then we have*

$$\mathcal{G}_{y_{6,q}}(t,x;k;\lambda,r)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\left[\begin{array}{c}k\\j\end{array}\right]_q^r \lambda^j\exp\left([x+j]_q t\right). \tag{15}$$

Let $\mathrm{i}^2=-1$. Substituting $t=\mathrm{i}\theta$ into (15) implies

$$\mathcal{G}_{y_{6,q}}(\mathrm{i}\theta,x;k;\lambda,r)=\frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\left[\begin{array}{c}k\\j\end{array}\right]_q^r \lambda^j\exp\left(\mathrm{i}[x+j]_q \theta\right). \tag{16}$$

By using the well-known Euler's formula:

$$\exp(\mathrm{i}\alpha)=\cos(\alpha)+\mathrm{i}\sin(\alpha)$$

in the equation (16) with the replacement α by $[x+j]_q\theta$, we get

$$\begin{aligned}\mathcal{G}_{y_{6,q}}(\mathrm{i}\theta,x;k;\lambda,r) &= \frac{1}{[k]_q!}\sum_{j=0}^{k}q^{\binom{j}{2}}\left[\begin{array}{c}k\\j\end{array}\right]_q^r \lambda^j \\ &\qquad\times\left(\cos\left([x+j]_q\theta\right)+\mathrm{i}\sin\left([x+j]_q\theta\right)\right).\end{aligned}$$

Considering real and imaginary parts of the above equation as a generating function, we define two new finite sums:

$$\begin{aligned}
\mathcal{G}_{C_{y6,q}}(\theta,x;k;\lambda,r) &:= \operatorname{Re}\{\mathcal{G}_{y6,q}(\mathrm{i}\theta,x;k;\lambda,r)\} \\
&= \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q^r \lambda^j \cos\left([x+j]_q\,\theta\right) \\
&= \sum_{k=0}^{\infty} C_{y6,q}(x;k;\lambda,r)\frac{\theta^k}{k!} \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_{S_{y6,q}}(\theta,x;k;\lambda,r) &:= \operatorname{Im}\{\mathcal{G}_{y6,q}(\mathrm{i}\theta,x;k;\lambda,r)\} \\
&= \frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q^r \lambda^j \sin\left([x+j]_q\,\theta\right). \\
&= \sum_{k=0}^{\infty} S_{y6,q}(x;k;\lambda,r)\frac{\theta^k}{k!}. \qquad (18)
\end{aligned}$$

By using the Taylor series expansion of the function $\cos\left([x+j]_q\,\theta\right)$ in (17), we get

$$\frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q^r \lambda^j \sum_{k=0}^{\infty}(-1)^k [x+j]_q^{2k}\frac{\theta^{2k}}{(2k)!} = \sum_{k=0}^{\infty} C_{y6,q}(x;k;\lambda,r)\frac{\theta^k}{k!}.$$

Comparing the coefficients of $\frac{\theta^k}{k!}$ on both sides of the above equation, we get the following theorem:

Theorem 2. *Let k be a nonnegative integer. Then we have*

$$C_{y6,q}(x;2k;\lambda,r) = \frac{(-1)^k}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q^r \lambda^j [x+j]_q^{2k}$$

and

$$C_{y6,q}(x;2k+1;\lambda,r) = 0.$$

By using the Taylor series expansion of the function $\sin\left([x+j]_q\,\theta\right)$ in (18), we get

$$\frac{1}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q^r \lambda^j \sum_{k=0}^{\infty}(-1)^k [x+j]_q^{2k+1}\frac{\theta^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} S_{y6,q}(x;k;\lambda,r)\frac{\theta^k}{k!}.$$

Comparing the coefficients of $\frac{\theta^k}{k!}$ on both sides of the above equation, we get the following theorem:

Theorem 3. *Let k be a nonnegative integer. Then we have*

$$S_{y6,q}(x;2k+1;\lambda,r) = \frac{(-1)^k}{[k]_q!}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q^r \lambda^j [x+j]_q^{2k+1}$$

and

$$S_{y6,q}(x;2k;\lambda,r) = 0.$$

Conclusion

As conclusion, this study presents two new finite sums involving higher-power of q-binomial coefficients together with sine and cosine functions. Future plan is to investigate these new two sums and provide their further properties and relationships with other special sums.

Acknowledgments

This paper is dedicated to Professor Taekyun KIM on the occasion of his 60th anniversary. For Professor KIM's great contribution to mathematics, I would like to present my endless respect to him and to congratulate his 60th birthday with my best wishes for a long and especially healthy life to him.

References

[1] L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. **15**, 987–1000, 1948.

[2] R. Goldman, P. Simeonov and Y. Simsek, *Generating functions for the q-Bernstein bases*, SIAM J. Discrete Math. **28 (3)**, 1009–1025, 2014.

[3] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Cambridge University Press, Cambridge, UK, 2005.

[4] V. Kac and P. Cheung, *Quantum calculus*, Springer-Verlag, New York, 2002.

[5] M.-S. Kim, D. Kim, The q-Stirling numbers of the second kind and its applications, J. Nonlinear Sci. Appl. **11**, 971–983, 2018.

[6] I. Kucukoglu, Remarks on a class of combinatorial numbers and polynomials, AIP Conf. Proc. **2849**, 2023; Article ID: 060004.

[7] I. Kucukoglu, *Analysis of a new class of q-combinatorial numbers and polynomials by its implementation in the Wolfram Language*, In: Proceeding Book of the 5th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2022) (Ed. by Y. Simsek, M. Alkan, I. Kucukoglu and O. Öneş), Antalya, Turkey, October 27-30, 2022, pp. 124–128; ISBN: 978-625-00-0917-8.

[8] I. Kucukoglu, *Computational implementations for analyzing the q-analogues of a combinatorial number family by their derivative formulas, interpolation functions and p-adic q-integrals*, Authorea 2023; DOI: 10.22541/au.168435038.81682594/v1.

[9] H. Ozden, I. N. Cangul and Y. Simsek, *Generalized q-Stirling numbers and their interpolation functions*, Axioms **2 (1)**, 10–19, 2013.

[10] Y. Simsek, *Computation methods for combinatorial sums and Euler type numbers related to new families of numbers*, Math. Methods Appl. Sci. **40 (7)**, 2347–2361 2017.

[11] Y. Simsek, *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials*, Appl. Anal. Discrete Math. **12**, 1–35, 2018.

[12] Y. Simsek, *Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers*, J. Math. Anal. Appl. **477**, 1328–1352, 2019.

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Interconnections and relationships among Fubini numbers and special numbers

Jongkyum Kwon

In this paper, we examine the interconnections between Fubini numbers and a range of recently defined special numbers, including the Lah-Bell numbers, the Changhee numbers, and the Dahee numbers, along with several other established special numbers. We explore these relationships by analyzing the generating functions associated with these numbers, which enables us to establish diverse formulae for Fubini numbers in relation to other special numbers. Additionally, we uncover new properties of Fubini numbers through this investigation. Furthermore, by employing formal power series of generating functions, we establish general identities for special numbers, thereby extending our understanding of their properties and relationships.

2020 MSC: 11B83, 11B68, 11B73

KEYWORDS: Fubini numbers, Lah numbers, Euler numbers, Changhee numbers, Lah-Bell numbers, Bell numbers, Frobenius-Euler numbers, Bernoulli numbers, Daehee numbers

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On the infinite series whose terms involve truncated lambda-exponentials

Sang Jo Yun [1], *Jongkyum Kwon* [2] *and Jin-Woo Park* [*3]

The degenerate logarithm functions which are the compositional inverse of the degenerate exponentials which were defined by Kim-Kim play an important role in studies on degenerate versions of many special numbers and polynomials. In this paper, we investigate infinite series whose terms involve truncated lambda-exponentials with binomial coefficients, the degenerate falling factorials, the Stirling numbers of the second kind and degenerate Stirling numbers of the second.

2020 MSC: 11B73, 11B65, 05A10

KEYWORDS: Stirling numbers, Special numbers and polynomials

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Probabilistic degenerate Bernoulli and degenerate Euler polynomials

Lingling Luo [*1], *Taekyun Kim* [2], *Dae San Kim* [3] *and Yuankui Ma* [4]

Recently, many authors have studied degenerate Bernoulli and degenerate Euler polynomials. Let Y be a random variable whose moment generating function exists in a neighborhood of the origin. We introduce and study the probabilistic extension of degenerate Bernoulli and degenerate Euler polynomials, namely the probabilistic degenerate Bernoulli polynomials associated with Y and the probabilistic degenerate Euler polynomials associated with Y. Also, we introduce the probabilistic degenerate r-Stirling numbers of the second kind associated with Y and the probabilistic degenerate two variable Fubini polynomials associated with Y. We obtain some properties, explicit expressions, recurrence relations and certain identities for those polynomials and numbers. As special cases of Y, we treat the gamma random variable with parameters $\alpha, \beta > 0$, the Poisson random variable with parameter $\alpha > 0$, and the Bernoulli random variable with probability of success p.

2020 MSC: 11B68, 11B83

KEYWORDS: Probabilistic degenerate Bernoulli polynomials, Probabilistic degenerate Euler polynomials, Probabilistic degenerate r-Stirling numbers of the second kind, Probabilistic degenerate two variables Fubini polynomials

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A study on the r-truncated Lah numbers and r-truncated Lah-Bell polynomials and their applications

Lee-Chae Jang

In this presentation, we define r-truncated Lah numbers and Lah-Bell polynomials, and investigate some useful identities of them. Furthermore, we obtain some related identities between the expected value of Poisson random variable and r-truncated Lah-Bell polynomials.

2020 MSC: 11B99

KEYWORDS: Lah numbers, Bell polynomials

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A note on degenerate poly-Bernoulli numbers and polynomials

Lee-Chae Jang

Kaneko [1] studied the poly-Bernoulli polynomials which are defined by using the polylogarithm functions. In this presentation, we study the degenerate poly-Bernoulli polynomials and numbers arising from polyexponential functions, and derive their explicit expressions and some identity involving them.

2020 MSC: 11B68, 11B83, 11B99

KEYWORDS: Bernoulli numbers and polynomials, polylogarithm functions, poly-exponential functions, special numbers and polynomials.

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References

[1] M. Kaneko, *poly-Bernoulli numbers*, J. Theor. Nombres Bordeaux **9 (1)**, 221–228, 1997.

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On a Jakimovski-Leviatan type operators defined via q-multiple Appell polynomials

Merve Çil [*1] *and Mehmet Ali Özarslan* [2]

In this paper, by using q-multiple Appell polynomials we introduce a form of Jakimovski-Leviatan type positive linear operators. We investigate convergence properties of our operators such as Krovkin-type approximation properties and calculate the rate of first order modulus of continuity, second order modulus of smoothness, Petree's K-functional and two kind of Lipschitz type space.

2020 MSC: 41A10, 41A25

KEYWORDS: Jakimovski-Leviathan operators, Szasz-Mirakyan operators, q-Calculus, q-Multiple Appell operators, Modulus of continuity, Weighted Korovkin theorem

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References

[1] A. Aral, *A generalization of Szász–Mirakyan operators based on q-integers*, Math. Comput. Modelling **47**, 1052–1062, 2008.

[2] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications*, Walter de Gruyter, 2011.

[3] A. Jakimovski and D. Leviatan, *Generalized Szász operators for the approximation in the infinite interval*, Mathematica (Cluj), **11 (34)**, 97–103, 1969.

[4] A. D. Gadjiev, *Theorems of the type of PP Korovkin type theorems*, Matematicheskie Zametki **20**, 781–786, 1976.

[5] A. D. O. Gadzhiev, *The convergence problem for a sequence of positive linear operators on unbounded sets, and theorems analogous to that of PP Korovkin*, Doklady Akademii Nauk Russian Academy of Sciences **5**, 1001–1004, 1974.

[6] N. I. Mahmudov, *On a class of q-Bernoulli and q-Euler polynomials*, Adv. Difference Equ. **2013 (1)**, 1–11, 2013.

[7] M. A. Özarslan, *Approximation properties of Jain-Appell operators*, Appl. Anal. Discrete Math. **14 (3)**, 654–669, 2020.

[8] M. A. Özarslan and O. Duman, *Local approximation results for Szász-Mirakjan type operators*, Arch. Math. **90**, 144–149, 2008.

[9] V. Kac and P. Cheung, *Quantum calculus*, Springer Science & Business Media, 2001.

[10] P. P. Korovkin, *On convergence of linear positive operators in the space of continuous functions*, Dokl. Akad. Nauk SSSR 961–964, 1953.

[11] D. Lee, *On multiple Appell polynomials*, Proc. Am. Math. Soc. **139 (6)**, 2133–2141, 2011.

[12] O. Szasz, *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Nat. Bur. Standards. **45**, 239–245, 1950.

[13] V. Totik, *Uniform approximation by Szász-Mirakjan type operators*, Acta Math. Hung. **41**, 291–3071, 1983.

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Prime submodule with respect to a multiplication set

Ortaç Öneş *[1] *and Mustafa Alkan* [2]

In this study, we define a prime submodule with respect to a multiplicative set, which generalizes the definition of S-prime submodule. We focus on the structure of prime submodules with respect to a multiplicative set under quotient module and module homomorphism.

2020 MSC: 16N60, 16U20

KEYWORDS: Prime ideal, Prime submodule, Multiplicative set

Introduction

Throughout this study, all rings are commutative with identity and all modules are unitary.
Let P be a submodule of M over a ring R. The ideal $(P : M)$ is defined as the set $\{r \in R : rM \subseteq P\}$. A proper submodule P of M is called prime if whenever $r \in R$, $m \in M$ and $rm \in P$, then $m \in P$ or $r \in (P : M)$. Let $Spec_R(M)$ be the set of all prime submodules of an R-module M. For example, $Spec_{\mathbb{Z}}(\mathbb{Z}) = \{0\} \cup \{p\mathbb{Z} : p$ is prime$\}$. As well known, prime submodules plays an important role to characterize modules and has been studied for long time by a lot of authors (*cf.* [1, 3, 5, 6]).
Let R be an integral domain and let M be an R-module.

$$T(M) = \{m \in M : \text{ there exists } 0 \neq r \in R \text{ such that } rm = 0\}$$

is a submodule of M. If $T(M) = 0$, M is called torsion free.

The following theorem gives a direct connection among prime ideal, prime submodule and torsion free module.

Theorem 1. *Let R be a ring and let M be an R-module. Then a submodule P of M is prime if and only if $(P : M)$ is a prime ideal of R and M/P is a torsion free R/P-module.*

A multiplicative set is a subset S of a ring R such that $1_R \in S$ and $xy \in S$ for all $x, y \in S$.
We define a prime submodule with respect to a multiplicative set, which is called SS-prime. This definition generalizes the definition of S-prime submodule. We examine quotient prime submodules with respect to a multiplicative set and its structure under module homomorphism.

Main results

Definition 2 (*cf.* [7])**.** *Let P be a submodule of M over a ring R and let S be a multiplicative set. If $rm \in P$ and $(P : M) \cap S = \emptyset$ implies $sr \in (P :_R M)$ or $sm \in P$ for any $r \in R$ and $m \in M$, P is called a S-prime submodule of M.*

We generalizes the above definition as follows.

Definition 3. *A submodule P of M over a ring R is called SS-prime if there are $s_1 \in S_1$ and $s_2 \in S_2$ such that $S_1 \subseteq S_2$, $rm \in P$ and $(P : M) \cap S_2 = \emptyset$ implies $s_1 r \in (P :_R M)$ or $s_2 m \in P$ for any $r \in R$ and $m \in M$. The set of all SS-prime submodules is denoted by $Spec_{SS}(M)$.*

Let S be a multiplicative set in R and let M be a module over R. $S^{-1}M$ is defined by the quotient module of M.

Proposition 4. *Let M be a module over a ring R. If P is a SS-prime submodule of R-module M, then $S_2^{-1}P$ is a SS-prime submodule of $S_1^{-1}R$-module $S_2^{-1}M$.*

Proof. Assume that P is prime submodule. Let $\dfrac{r}{s_1} \in S_1^{-1}R$ and $\dfrac{m}{s_2} \in S_2^{-1}M$ such that $\dfrac{rm}{s_1 s_2} \in S_2^{-1}P$. Then there exists the element s_2^* in S_2 such that $s_2^* rm \in P$ and it follows $s_2^{**} m \in P$ or $s_1^* s_2^* r \in (P : M)$ with assumption. Then we have that $\dfrac{m}{s_2} = \dfrac{s_2^{**} m}{s_2^{**} s_2} \in S_2^{-1}P$ or $\dfrac{r}{s_1} = \dfrac{s_1^* s_2^* r}{s_1^* s_2^* s_1} \in S_2^{-1}(P : M) \subseteq \left(S_2^{-1}P : S_2^{-1}M\right)$, implying that $S_2^{-1}P$ is SS-prime submodule. □

Proposition 5. *Let M_1 and M_2 be modules over a ring R and let $f : M_1 \to M_2$ be an R-module homomorphism. If*

$$P_2 \in Spec_{SS}(M_2) \quad \textit{and} \quad \left(f^{-1}(P_2) : M_1\right) \cap S_2 = \emptyset,$$

then

$$f^{-1}(P_2) \in Spec_{SS}(M_1).$$

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References

[1] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Comm. Algebra **30 (9)**, 4407–4416, 2022.

[2] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing, London, 1969.

[3] A. Azizi, *Radical formula and prime submodule*, J. Algebra **307**, 454–460, 2007.

[4] D. S. Dummit and R. M. Foote, *Abstract algebra*, Prentice Hall, New Jersey, 1999.

[5] C. P. Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Pauli **33 (1)**, 61–69, 1984.

[6] R. L. McCasland and P. F. Smith, *Prime submodules of noetherian modules*, Rocky Mountain J. Math. **23 (3)**, 1041–1062, 1993.

[7] E. Ş. Sevim, T. Arabacı, Ü. Tekir and S. Koç, *On S-prime submodules*, Turkish J. Math. **43 (2)**, 1036–1046, 2019.

[8] O. Öneş and M. Alkan, *Multiplication modules with prime spectrum*, Turkish J. Math. **43 (4)**, 2000–2009, 2019.

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On the relation between palindrome compositions and Fibonacci numbers

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In this note, we deal with the composition sets and the palindrome composition sets. Then we define polindrome color compositions and get the integer sequence for the numbers of polindrome color compositions.

2020 MSC: 05A15, 05A17, 05A18, 11B39, 11B99

KEYWORDS: Compositions of the integers, The n-color compositions of the integers, Palindromes compositions, Fibonacci numbers, Generating function

Introduction

In literature, one can find concepts that are presented in a straightforward manner or clearly defined, which possess significant characteristics and can be applied in various contexts. In the realm of mathematics, both Fibonacci numbers and compositions of a positive integer are examples of such concepts. Since the establishment of these concepts, they have garnered the interest of numerous scientists, and their findings, which have yielded significant ramifications, have made remarkable contributions to nearly all scientific disciplines. These findings have resulted in the advancement of mathematical analysis and number theory.

The Fibonacci numbers denoted by f_n are that each number is the sum of the two preceding ones, starting from 0 and 1. That is, $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. The intricate relationship between the Fibonacci numbers and the golden ratio is deeply intertwined with various disciplines of science and technology. Likewise, the extrapolations of Fibonacci numbers and the other distinctive numbers are employed in numerous mathematical frameworks.

A composition of an integer n is a representation of n as a sum of positive integers. In the combinatorics, a classical result about the number of compositions of n with an integer k parts is given by the coefficient of x^n of the polynomial or power series $\left(\sum\limits_{i=1}^{\infty} x^i\right)^k$ where $|x| < 1$. These coefficients exhibit fascinating mathematical properties, closely resembling Binomial coefficients and have many useful applications (*cf.* [1, 2, 10, 15, 19, 20, 21, 26]).

In the study conducted by Hoggart and Lind, as documented in the publication by Hoggatt [21], an examination was made into the correlation between the composition of an integer and Fibonacci numbers. This investigation utilized Binomial properties and resulted in a proof.

(i) f_n is the number compositions of an integer n into odd parts

(ii) f_{2n} is the sum of the products of the parts over all compositions of an integer n, i.e.

$$f_{2n} = \sum_{a_1+a_2+...+a_k=n} a_1 a_2 ... a_k. \tag{1}$$

Recently, there has been interested n-color compositions of an integer m is defined as composition of m for which a part of size n can take on n color (*cf.* [1, 2, 26]). Then by the identity equation (1), it is clear that the number of n-color compositions of an integer m is f_{2m} the $2m$th Fibonacci number.

The composition $(1,2)$ of 3 is colored as

$$(1_1, 2_1), (1_1, 2_2),$$

and so the color composition of 3

$$(3_1), (3_2), (3_3), (1_1, 2_1,), (1_1, 2_2), (2_1, 1_1), (2_2, 1_1), (1_1, 1_1, 1_1).$$

There are twenty one n-color compositions of 4:

$$\begin{aligned}
&(4_1), (4_2), (4_3), (4_4), (1_1, 3_1,), (1_1, 3_2), (1_1, 3_3), (3_1, 1_1), (3_2, 1_1),\\
&(3_3, 1_1), (2_1, 2_1), (2_1, 2_2), (2_2, 2_1), (2_2, 2_2), (2_1, 1_1, 1_1), (1_1, 2_1, 1_1),\\
&(1_1, 1_1, 2_1), (2_2, 1_1, 1_1), (1_1, 2_2, 1_1), (1_1, 1_1, 2_2), (1_1, 1_1, 1_1, 1_1).
\end{aligned}$$

A palindrome composition is a composition whose part-sequence is the same whether it is read from left to right to right to left. In [18], it is shown that are $2^{\|\frac{n}{2}\|}$ palindromes of an integer n and also in [26], Shapcott investigated the formula for the number of color palindromes of an integer n.

We denote the composition set of an integer n as follows:

$$P_n = \left\{(a_1, a_2, ..., a_t) : \; a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\}.$$

Then from [4], we recall the following operations for the element $a = (a_1, a_2, ..., a_t) \in P_n$ and an integer j;

$$\begin{aligned}
(j \odot a) &= (j, a_1, a_2, ..., a_t),\\
(j \oplus a) &= (a_1 + j, a_2, ..., a_t).
\end{aligned}$$

Then we use the notations $j \oplus P_n$ and $j \odot P_n$ for the following sets

$$\begin{aligned}
j \oplus P_n &= \{l \oplus a : a \in P_n\},\\
j \odot P_n &= \{l \odot a : a \in P_n\}.
\end{aligned}$$

Theorem 1 (*cf.* [4])**.** *Let n, r be positive integers $(r \leq n)$. Then the set P_n is disjoint union of the sets*

$$(r \oplus P_{n-r}) \quad \textit{and} \quad (i \odot P_{n-i})$$

for all $i \in \{1, \ldots, r\}$, i.e.

$$P_n = (\cup_{i=1}^{r}(i \odot P_{n-i}) \cup (r \oplus P_{n-r}).$$

Proof. It is sufficient to prove the inclusion $P_n \subseteq (\cup_{i=1}^{r}(i \odot P_{n-i}) \cup (r \oplus P_{n-r})$.

Let $x = (a_1, \ldots, a_m) \in P_n$. If $a_1 \leq r$ then $x \in \cup_{i=1}^{r}(i \odot P_{n-i})$.

Now assume that $r < a_1$. Then $b = a_1 - r$ and so define the element $y = (b, a_2, a_3, \ldots, a_m) \in P_{n-r}$. Then it is clear that $x = r \oplus y \in (r \oplus P_{n-r})$.

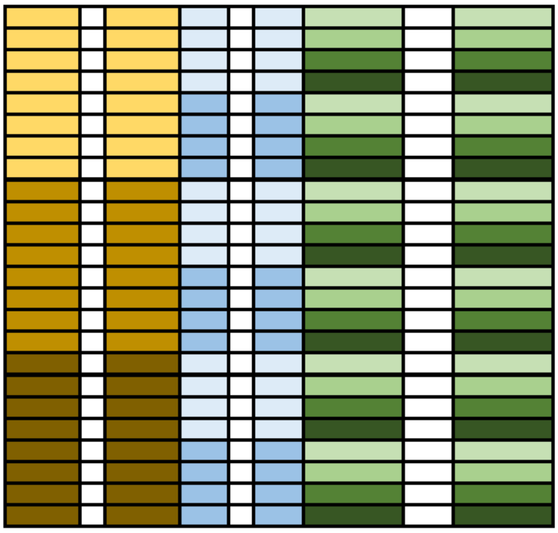
It is also clear that $(r \oplus P_{n-r}) \cap (i \odot P_{n-i}) = \varnothing$ for all $i \in \{1, \ldots, r\}$. □

Then we also get the following result in [4, Theorem 5].

Corollary 2 (*cf.* [4])**.** *For a positive integer n, we have*

$$P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n).$$

The positive integer n greater than 3 can be represented as either $x = (c, 2, c)$ or $x = (c, 1, c)$, where c is an integer and the parts with size c can be colored in c different colors, while the middle parts can be colored in white. Subsequently, the arrangement of x exhibits a symmetrical pattern in terms of colors, and we designate the color that forms the palindrome in the segment x. The color composition of an integer that consists of palindrome color parts is referred to as the palindrome color composition of the integer. Subsequently, we can depict the arrangement of the palindrome color (7,5,10) in the subsequent manner:



Then we compute the sequence for the palindrome color compositions

$$1, 2, 4, 8, 17, 35, 73, 151, ...$$

We wonder about determining the recurrence relation and generating functions for this sequence, as well as exploring any relationships it may have with Fibonacci numbers.

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References

[1] A. K. Agarval, *n-colour composition*, Indian J. Pure Appl. Math. **31 (11)**, 1421–1427, 2000.

[2] A. K. Agarwal and G. E. Andrews, *Rogers-Ramanujan identities for partitions with "n copies of n"*, J. Combin. Theory Ser. A. **45 (1)**, 40–49, 1987.

[3] B. Al and M. Alkan, *On relations for the partitions of numbers*, Filomat **34 (2)**, 567–574, 2020.

[4] B. Al and M. Alkan, *Some relations between partitions and Fibonacci numbers*, In: Proceedings Book of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM2019) (Ed. by Y. Simsek, A. Bayad, M. Alkan, I. Kucukoglu and O. Ones), Paris, France, August 28-31, 2019, pp. 14–17; ISBN: 978-2-491766-00-9.

[5] G. E. Andrews, *The theory of partitions*, Addison-Wesley Publishing, New York, 1976.

[6] G. E. Andrews and K. Erikson, *Integer partitions*, Cambridge University Press, Cambridge, 2004.

[7] G. E. Andrews, M. D. Hirschhorn and J. A. Sellers, *Arithmetic properties of partitions with even parts distinct*, Ramanujan J. **23 (1–3)**, 169–181, 2010.

[8] T. M. Apostol, *On the Lerch zeta function*, Pacific J. Math. **1**, 161–167, 1951.

[9] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976.

[10] M. Archibald, A. Blecher and A. Knopfmacher, *Inversions and parity in compositions of integers*, J. Integer Seq. **23**, 2020; Article ID: 20.4.1.

[11] D. Birmajer, J. B. Gil and M. M. D. Weiner *$(an + b)$-color compositions*, 2017; ArXiv:1707.07798.

[12] S. C. Chen, *On the number of partitions with distinct even parts*, Discrete Math. **311**, 940–943, 2011.

[13] L. Euler, *Introduction to analysis of the infinite* (Volume 1), (Translation by J. D. Blanton), Springer-Verlag, 1988.

[14] J. A. Ewell, *Recurrences for the partition function and its relatives*, Rocky Mountain J. Math. **34 (2)**, 619–627, 2004.

[15] I. M. Gessel and J. Li, *Compositions and Fibonacci identities*, J. Integer Seq. **16**, 2013; Article ID: 13.4.5.

[16] B. Gil and J. A. Tomosko, *Fibonacci clored compostions and applications*, 2021; ArXiv:2108.06462.

[17] H. Gupta, *Partitions-A survey*, Journal of Research of the Notional Bureau of Standards - B. Mathematical Sciences, 74B, 1, 1970.

[18] S. Heubach, P. Z. Chinn and R. P. Grimaldi, *Rises levels, Drops and "+" signs in compositions: Extentions of a paer by Alladi and Hoggatt*, Fibonacci Quart. **41 (3)**, 229–239, 2003.

[19] S. Heubach and T. Mansour, *Compositions of n with parts in a set*, Congr. Numer. **164**, 127–143, 2004.

[20] S. Heubach and T. Mansour, *Combinatorics of compositions and words*, CRC Press, 2010.

[21] V. E. Hoggatt and D. A. Lind, *Fibonacci and binomial properties of weighted compositions*, Jour. Comb. Theo. **4**, 121–124, 1968.

[22] A. F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart. **34 (1)**, 40–54, 1996.

[23] M. Janjic, *Some formulas for numbers of restricted words*, J. Integer Seq. **20**, 2017; Article ID: 17.6.5.

[24] T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley-Interscience Publication, Canada, 2001.

[25] H. Merzouka, A. Boussayoudb and M. Chelgham, *Generating functions of generalized tribonacci and tricobsthal polynomials*, Montes Taurus J. Pure Appl. Math. **2 (2)**, 7–37, 2020.

[26] C. Shapcott, *C-color compositions and palindromes*, Fibonacci Quart. **50 (4)**, 297–303, 2012.

[27] R. P. Stanley, *Enumerative combinatorics* (Volume 1; 2nd Edition), Cambridge University Press, 2011.

[28] Y. Simsek, *Generating Funcstions for finite sums involving higher powers of binomial coeffients: Analysis of hypergeometric functions inculudinf new families of polynomilies and numbers*, J. Math. Anal. Appl. **477**, 2328–1352, 2019.

[29] G. Ozdemir, Y. Simsek and G. V. Milovanovic, *Generating funcstions for specials polynomials and numbers including Apostos-type and Humbert-type polynomials*, Mediterr. J. Math. **14**, 2017; Article ID: 117.

[30] H. S. Wilf, *Generating functionology*, Academic Press, Inc., 1994.

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New type Szász-Mirakyan operators

Mustafa Kara

Approximation theory, which has great application potential, is one of the significant research topics of mathematical analysis and has been studied by many mathematicians around the world. One of the strongest aspects of approximation theory is that it sheds light on many scientific problems in other fields, especially basic sciences and engineering sciences. One of the most important of these is Szász's generalization of Bernstein polynomials to infinite intervals as classical Szász-Mirakyan operators in 1950 as follows:

$$S_n(f;x) = ne^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{x^k n^k}{k!}, \qquad x \in [0,\infty), \qquad n \in \mathbb{N},$$

for the function $f \in [0,\infty)$.

In this paper, we introduce new generalazition of of Szász-Mirakjan operators

$$M_n^*(f;x) := \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$s_{n,k}(x) = ne^{-n\tau} \frac{x^{k-1} n^k}{k!} \left(\frac{k}{n} - x\right)^2.$$

Respectively, the local approximation properties of these operators are given through Peetreís K-function and rate of convergence by utilizing the ordinary of modulus of continuity and lipschitz type maximal functions are studied.

2020 MSC: 41A25, 41A36, 47A58

KEYWORDS: Szász–Mirakjan operators, Moments

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References

[1] S. Bernstein, *Demonstration du theoreme de Weierstrass, fondee sur le calculus des piobabilitts*, Commun. Soc. Math. Kharkow **13**, 1–2, 1913.

[2] H. Bohman, *On approximation of continuous and analytic functions*, Ark. Math. **2**, 43–46, 1952.

[3] F. Cheng, *On the rate of convergence of the Szasz-Mirakyan operator for functions of bounded variation*, J. Approx. Theory **40**, 226–241, 1984; https://doi.org/10.1016/0021-9045(84)90064-9.

[4] O. Duman and M. A. Özarslan, *Szász-Mirakjan type operators providing a better error estimation*, Appl. Math. Lett.**20**, 1184–1188, 2007.

[5] O. Duman, M. A. Özarslan and H. Aktuglu, *Better error estimation for Szász-Mirakjan-Beta operators*, J. Comput. Anal. Appl. **10**, 53–59, 2008.

[6] O. Duman, M. A. Özarslan and B. D. Vecchia, *Modified Szász-Mirakjan-Kantorovich operators preserving linear functions*, Turkish J. Math. **33**, 151–158, 2009.

[7] V. Gupta and R. P. Pant, *Rate of convergence for the modified Szász-Mirakjan operators on bounded variation*, J. Math. Anal. Appl. **233**, 476–483, 1999.

[8] N. Ispir and C. Atakut, *Approximation by modified Szász-Mirakjan operators on weighted spaces*, Proc. Indian Acad. Sci. (Math. Sci.) **112**, 571–578, 2002.

[9] P. P. Korovkin, *Convergence of linear positive operators in the spaces of continuous functions* (Russian), Doklady Akad. Nauk.SSSR (N.S.) **90**, 961–964, 1953.

[10] O. Szasz, *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Nat. Bur. Stand. **45**, 239–245, 1950.

[11] L. Rempulska and S. Graczyk, *Approximation by modified Szász-Mirakjan operators*, J. Inqual. Pure Appl. Math. **10 (3)**, 2009; Article ID: 61.

[12] K. G. Weierstrass, *Über die analyishe Darstellbar keit sogenonnter willk ürlincher Funtionen einer reelen Ver ünderli*, Sitz. Akad. Berl. **2**, 633–693, 1885.

[13] D. X. Zhou, *Weighted approximation by Szasz-Mirakjan operators*, J. Approx. Theory **76**, 393–402, 1994; https://doi.org/10.1006/jath.1994.1025.

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A note on a Voronovskaya-type relation for positive linear operators based on Bernoulli-type polynomials

Mine Menekşe Yılmaz

One of the main problems in approximation theory is estimating the rate of convergence of sequences of positive linear operators. Voronovskaya-type formulae are one of the most useful tools for studying this. In 1932, [13], Voronovskaya gave a relation for Bernstein polynomials, [5], that if $f \in C^2([0,1])$, then the following equality holds:

$$\lim_{n\to\infty} n\left[\tilde{B}_n(f;x) - f(x)\right] = \frac{x(1-x)}{2!} f''(x).$$

This relation was later studied for some other linear positive operators by [4, 6, 7, 8, 9]. In constructing a Voronovskaya-type formula for a sequence of operators, some properties of the operators, such as central moments, need to be known. That is why the central moments of the studied operator will be mentioned in this study. For investigation we chose an operator containing the Bernoulli-type polynomial whose definition is given in [12]. Based on the concepts mentioned, it will be presented that the Voronovskaya type formula is valid for an operator sequence containing Bernoulli-type polynomials. For some articles on certain polynomials or operators that are based on mentioned polynomials, we recommend articles [1, 2, 3, 10, 11].

2020 MSC: 41A10, 41A25, 41A36

KEYWORDS: Voronovskaya type theorem, Generating functions

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References

[1] E. Agyuz, *Identities derived from a particular class of generating functions for Frobenius-Euler type Simsek numbers and polynomials*, Filomat **38 (5)**, 1531–1545, 2024.

[2] E. Agyuz, *Convergence Properties of a Kantorovich Type of Szász Operators Involving Negative Order Genocchi Polynomials*, Gazi University Journal of Science Part A: Engineering and Innovation **10 (2)**, 196–205, 2023.

[3] P. Appell, *P. Sur une classe de polynômes*, Ann. Sci. Éc. Norm. Supér.; Société mathématique de France, Paris, France, **2 (9)**, 119–144, 1880.

[4] D. Barbosu, *Kantorovich-Stancu type operators*, J. Inequal. Pure Appl. Math. **5 (3)**, 1–6, 2004.

[5] S. N. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*, Commun. Kharkov Math. Soc. **13**, 1–2, 1912/1913.

[6] P. L. Butzer and R. J. Nessel, *Fourier analysis and approximation* (Volume I), Birkhauser, Bessel and Academic Press, New York, 1971.

[7] N. Cetin and V. A. Radu, *Approximation by generalized Bernstein-Stancu operators*, Turk. J. Math. **43 (4)**, 2032–2048, 2019.

[8] D. Popa, *Intermediate Voronovskaja type results for the Lototsky–Bernstein type operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM **114 (1)**, 2020; Article ID: 12.

[9] L. Rempulska and M. Skorupka, *The Voronovskaya theorem for some operators of the Szasz-Mirakjan type*, Le Matematiche **50 (2)**, 251–261, 1995.

[10] P. Natalini and E. Ricci, *Adjoint Appell-Euler and first kind Appell-Bernoulli polynomials*, Appl. Appl. Math. **14 (2)**, 1112–1122, 2019.

[11] Y. Simsek, *Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions*, Adv. Stud. Contemp. Math. **16 (2)**, 251–278, 2008.

[12] M. M. Yilmaz, *Rate of convergence by Kantorovich type operators involving adjoint Bernoulli polynomials*, Publ. Inst. Math., Nouv. Sér. **114 (128)**, 51–62, 2023.

[13] E. Voronovskaya, *Determination de la forme asymptotique d'approximation des fonctions par les polynomes de S. N. Bernstein*, Dokl. Akad. Nauk SSSR Ser. A. 79–85, 1932.

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Incomplete Hermite Appell polynomials and their properties

Mehmet Ali Özarslan

The purpose of this study is to introduce the incomplete Hermite Appell polynomials and investigate their certain properties such as recurrence relations, lowering and raising operators, differential equations. Moreover, we derive several families of multilinear and mixed multilateral finite series relations and generating functions for the incomplete Hermite Appell polynomials.

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KEYWORDS: Hermite Appell polynomials, Operator

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A characterization of rotational minimal surface in dual Lorentzian space

Sousan Latifinavid [1], *Nemat Abazari* [*2] *and Yusuf Yayli* [3]

In this paper, we have considered a variation problem that its solution is related to the generating curves of rotational surface in the dual Lorentzian space. It is shown that these curves have similar properties as a center of mass with some curves of space. Also, it is shown that this characterization of rotational minimal surface in the dual Lorentzian space is an extension of the known properties of the catenary and the catenoid in the Euclidean space.

2020 MSC: 55R50

KEYWORDS: Dual number, Dual vector space, Dual Lorentzian space, Spacelike surface, Timelike surface, Minimal surface, Rotational surface

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On generating functions for parametric kinds of unified and modified presentation of Fubini polynomials

Neslihan Kilar

The main goal of this paper is to examine the unified and modified presentation of Fubini polynomials and their parametric kinds. The relations between these polynomials and other special polynomials are given with the help of their generating functions. Finally, some special cases are also given for the Fubini type numbers and polynomials.

2020 MSC: 05A19, 11C08, 11B83

KEYWORDS: Generating functions, Fubini type numbers and polynomials, Unified and modified presentation of Fubini numbers and polynomials, Special polynomials

Introduction

Special numbers, polynomials, and functions are used in many areas of mathematics and computer modeling. One of them, the Fubini numbers, are widely applications in number theory and enumerative combinatorics. These numbers count the number of weak orderings on a set of n elements. In addition to these, enumerations of certain trees, Cayley permutations, ordered multiplicative partitions of square free numbers, and the outcome of a horse race are also used. Many researchers have been studied different generalizations and parametric kinds of the Fubini numbers (*cf.* [1]- [22]). With the help of generating function methods, we investigate some of their properties of these numbers and polynomials and give some novel formulas.

The generating function of Fubini numbers is given by

$$\frac{1}{2-e^{t}}=\sum_{n=0}^{\infty} w_{g}\left(n\right)\frac{t^{n}}{n!} \tag{1}$$

(*cf.* [2, 3]).

The generating functions of Fubini type numbers and polynomials of order r are given by

$$\frac{2^{r}}{\left(2-e^{t}\right)^{2r}}=\sum_{n=0}^{\infty} a_{n}^{(r)}\frac{t^{n}}{n!} \tag{2}$$

and

$$\frac{2^{r}e^{yt}}{\left(2-e^{t}\right)^{2r}}=\sum_{n=0}^{\infty} a_{n}^{(r)}\left(y\right)\frac{t^{n}}{n!} \tag{3}$$

(*cf.* [9]).

When $r=1$ in (1), the following well-known relation is given (*cf.* [9]):

$$a_{n}^{(1)}=\sum_{p=0}^{n}\binom{n}{p}w_{g}\left(p\right)w_{g}\left(n-p\right).$$

The generating functions of polynomials $C_n(y,u)$ and $S_n(y,u)$ are given by

$$e^{yt}\cos(ut)=\sum_{n=0}^{\infty}C_n(y,u)\frac{t^n}{n!} \tag{4}$$

and

$$e^{yt}\sin(ut)=\sum_{n=0}^{\infty}S_n(y,u)\frac{t^n}{n!} \tag{5}$$

(*cf.* [19]). Note that the polynomials $C_n(y,u)$ and $S_n(y,u)$ are the Taylor-Maclaurin expansions of the two functions (*cf.* [12, 14, 15, 18, 19, 21]; see also the references cited therein).

Recently, Kilar and Simsek [15] defined the generating functions for unified and modified presentation of Fubini numbers and polynomials of order r, respectively:

$$\frac{2^r}{(\mu b^t-\vartheta)^{2r}}=\sum_{n=0}^{\infty}\mathfrak{a}_n^{(r)}(\mu;\vartheta,b)\frac{t^n}{n!} \tag{6}$$

and

$$\frac{2^r}{(\mu b^t-\vartheta)^{2r}}d^{tx}=\sum_{n=0}^{\infty}\mathfrak{a}_n^{(r)}(x,\mu;\vartheta,b,d)\frac{t^n}{n!}, \tag{7}$$

where $b,d\in\mathbb{R}^+$ with $b,d\geq 1$, $\vartheta,\mu,t\in\mathbb{C}$ and $\mu\neq\vartheta$, $|t|<\frac{2\pi}{|\ln b|}$ when $\mu=\vartheta$; $\left|t\ln b+\ln\left(\frac{\mu}{\vartheta}\right)\right|<2\pi$ when $\mu\neq\vartheta$; $1^r:=1$.

Using (7), Kilar and Simsek also defined the parametric kinds of unified and modified presentation of Fubini polynomials of order r:

$$\frac{2^r e^{ty\ln d}}{(\mu b^t-\vartheta)^{2r}}\cos(tu\ln d)=\sum_{n=0}^{\infty}\mathfrak{a}_n^{(r,C)}(y,u,\mu;\vartheta,b,d)\frac{t^n}{n!} \tag{8}$$

and

$$\frac{2^r e^{ty\ln d}}{(\mu b^t-\vartheta)^{2r}}\sin(tu\ln d)=\sum_{n=0}^{\infty}\mathfrak{a}_n^{(r,S)}(y,u,\mu;\vartheta,b,d)\frac{t^n}{n!} \tag{9}$$

(*cf.* [15]).

As a special case when $b=d=e$, $\mu=1$, $\vartheta=2$, (8) and (9) reduced to the two parametric kinds of Fubini-type polynomials:

$$\mathfrak{a}_n^{(r,C)}(y,u,1;2,e,e)=a_n^{(r,C)}(y,u)$$

and

$$\mathfrak{a}_n^{(r,S)}(y,u,1;2,e,e)=a_n^{(r,S)}(y,u)$$

and also the generating functions of polynomials $a_n^{(r,C)}(y,u)$ and $a_n^{(r,S)}(y,u)$ are given by

$$\frac{2^r e^{ty}}{(e^t-2)^{2r}}\cos(tu)=\sum_{n=0}^{\infty}a_n^{(r,C)}(y,u)\frac{t^n}{n!}$$

and

$$\frac{2^r e^{ty}}{(e^t-2)^{2r}}\sin(tu)=\sum_{n=0}^{\infty}a_n^{(r,S)}(y,u)\frac{t^n}{n!}$$

(*cf.* [21]).

Main results

In this section, by making use of the generating function methods, we obtain some formulas for the polynomials $\mathfrak{a}_n^{(r,C)}(y,u,\mu;\vartheta,b,d)$ and the polynomials $\mathfrak{a}_n^{(r,S)}(y,u,\mu;\vartheta,b,d)$, some special polynomials and the Fubini type numbers.

Theorem 1. *For $n \in \mathbb{N}_0$, we have*

$$\mathfrak{a}_n^{(r,C)}(y,u,\mu;\vartheta,b,d) = \sum_{p=0}^{n} \binom{n}{p} \mathfrak{a}_p^{(r)}(\mu;\vartheta,b)\, C_{n-p}(y\ln d, u\ln d). \tag{10}$$

Proof. From (4), (6) and (8), we can write

$$\sum_{n=0}^{\infty} \mathfrak{a}_n^{(r,C)}(y,u,\mu;\vartheta,b,d)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{a}_n^{(r)}(\mu;\vartheta,b)\frac{t^n}{n!} \sum_{n=0}^{\infty} C_n(y\ln d, u\ln d)\frac{t^n}{n!}.$$

Thus

$$\sum_{n=0}^{\infty} \mathfrak{a}_n^{(r,C)}(y,u,\mu;\vartheta,b,d)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \binom{n}{p} \mathfrak{a}_p^{(r)}(\mu;\vartheta,b)\, C_{n-p}(y\ln d, u\ln d)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of this last equation, we arrive at Theorem 1. □

Combining the following relation

$$C_n(kz, k\omega) = k^n C_n(z,\omega)$$

(*cf.* [12]) with (10), we have the following corrollary:

Corollary 2. *For $n \in \mathbb{N}_0$, we have*

$$\mathfrak{a}_n^{(r,C)}(y,u,\mu;\vartheta,b,d) = \sum_{p=0}^{n} \binom{n}{p} (\ln d)^{n-p}\, \mathfrak{a}_p^{(r)}(\mu;\vartheta,b)\, C_{n-p}(y,u).$$

Theorem 3. *For $n \in \mathbb{N}_0$, we have*

$$\mathfrak{a}_n^{(r,S)}(y,u,\mu;\vartheta,b,d) = \sum_{p=0}^{n} \binom{n}{p} \mathfrak{a}_p^{(r)}(\mu;\vartheta,b)\, S_{n-p}(y\ln d, u\ln d). \tag{11}$$

Proof. By (5), (6) and (9), we obtain

$$\sum_{n=0}^{\infty} \mathfrak{a}_n^{(r,S)}(y,u,\mu;\vartheta,b,d)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{a}_n^{(r)}(\mu;\vartheta,b)\frac{t^n}{n!} \sum_{n=0}^{\infty} S_n(y\ln d, u\ln d)\frac{t^n}{n!}.$$

Hence

$$\sum_{n=0}^{\infty} \mathfrak{a}_n^{(r,S)}(y,u,\mu;\vartheta,b,d)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \binom{n}{p} \mathfrak{a}_p^{(r)}(\mu;\vartheta,b)\, S_{n-p}(y\ln d, u\ln d)\frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of this last equation, we arrive at Theorem 3. □

From

$$S_n(kz, k\omega) = k^n S_n(z, \omega)$$

(*cf.* [12]) and (11), we obtain the following corrollary:

Corollary 4. *For $n \in \mathbb{N}_0$, we have*

$$\mathfrak{a}_n^{(r,S)}(y, u, \mu; \vartheta, b, d) = \sum_{p=0}^{n} \binom{n}{p} (\ln d)^{n-p}\, \mathfrak{a}_p^{(r)}(\mu; \vartheta, b)\, S_{n-p}(y, u).$$

When $b = d = e$ $\mu = 1$ $\vartheta = 2$ in (10) and (11), we get the following corrollary given by Srivastava and Kızılateş [21].

Corollary 5. *For $n \in \mathbb{N}_0$, we have*

$$a_n^{(r,C)}(y, u) = \sum_{p=0}^{n} \binom{n}{p} a_p^{(r)} C_{n-p}(y, u)$$

and

$$a_n^{(r,S)}(y, u) = \sum_{p=0}^{n} \binom{n}{p} a_p^{(r)} S_{n-p}(y, u)$$

(cf. [21]*).*

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References

[1] N. G. Acala, *A unification of the generalized multiparameter Apostol-type Bernoulli, Euler, Fubini, and Genocchi polynomials of higher order*, Eur. J. Pure Appl. Math. **13 (3)**, 587–607, 2020.

[2] L. Comtet, *Advanced combinatorics: The art of finite and infinite expansions*, D. Reidel Publishing Company, Dordrecht-Holland, Boston, 1974.

[3] I. J. Good, *The number of ordering of n candidates when ties are permitted*, Fibonacci Quart. **13**, 11–18, 1975.

[4] N. Kilar, *Fubini type numbers and their generating functions*, Akdeniz University, MSc Thesis in Mathematics, Antalya, 2017.

[5] N. Kilar, *Remarks and some formulas on Fubini type numbers and polynomials*, In: Proceedings Book of the 3rd & 4th Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020-2021) (Ed. by Y. Simsek, M. Alkan, I. Kucukoglu and O. Öneş), Antalya, Turkey, November 11-12, 2021, pp. 45–49; ISBN: 978-625-00-0397-8.

[6] N. Kilar, *Combinatorial sums and identities associated with functional equations of generating functions for Fubini type polynomials*, GUJ Sci **36 (2)**, 807–817, 2023.

[7] N. Kilar, *Formulas for Fubini type numbers and polynomials of negative higher order*, Montes Taurus J. Pure Appl. Math. **5 (3)**, 23–36, 2023.

[8] N. Kilar, *Notes on Fubini type numbers and their properties*, Accepted for the publication in AIP Conf. Proc. 2023.

[9] N. Kilar and Y. Simsek, *A new family of Fubini numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials*, J. Korean Math. Soc. **54 (5)**, 1605–1621, 2017.

[10] N. Kilar and Y. Simsek, *Identities and relations for Fubini type numbers and polynomials via generating functions and p-adic integral approach*, Publ. Inst. Math., Nouv. Sér. **106 (120)**, 113–123, 2019.

[11] N. Kilar and Y. Simsek, *Some relationships between Fubini type polynomials and other special numbers and polynomials*, AIP Conf. Proc. 2116, 2019; Article ID: 100017.

[12] N. Kilar and Y. Simsek, *Combinatorial sums involving Fubini type numbers and other special numbers and polynomials: Approach trigonometric functions and p-adic integrals*, Adv. Stud. Contemp. Math. **31 (1)**, 75–87, 2021.

[13] N. Kilar and Y. Simsek, *Formulae to Fubini type numbers emerge from application of p-adic integrals*, GU J Sci, Part A **8 (4)**, 402–410, 2021.

[14] N. Kilar and Y. Simsek, *Computational formulas and identities for new classes of Hermite-based Milne–Thomson type polynomials: Analysis of generating functions with Euler's formula*, Math. Methods Appl. Sci. **44**, 11245–11268, 2021.

[15] N. Kilar and Y. Simsek, *Families of unified and modified presentation of Fubini numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **5 (1)**, 1–21, 2023.

[16] T. Kim, D. S. Kim and G.-W. Jang, *A note on degenerate Fubini polynomials*, Proc. Jangjeon Math. Soc. **20 (4)**, 521–531, 2017.

[17] D. S. Kim, T. Kim, H.-I. Kwon and J.-W. Park, *Two variable higher-order Fubini polynomials*, J. Korean Math. Soc. **55 (4)**, 975–986, 2018.

[18] T. Kim and C. S. Ryoo, *Some identities for Euler and Bernoulli polynomials and their zeros*, Axioms **7 (3)**, 2018; Article ID: 56.

[19] M. Masjed-Jamei and W. Koepf, *Symbolic computation of some power-trigonometric series,* J. Symbolic Comput. **80,** 273–284, 2017.

[20] Y. Simsek, *Explicit formulas for p-adic integrals: Approach to p-adic distributions and some families of special numbers and polynomials*, Montes Taurus J. Pure Appl. Math. **1 (1)**, 1–76, 2019.

[21] H. M. Srivastava and C. Kızılateş, *A parametric kind of the Fubini-type polynomials*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**, 3253–3267, 2019.

[22] H. M. Srivastava, R. Srivastava, A. Muhyi, G. Yasmin, H. Islahi and S. Araci, *Construction of a new family of Fubini-type polynomials and its applications*, Adv. Difference Equ. **2021**, 2021; Article ID: 36.

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The second omega coindex

Nurten Urlu Ozalan

It has been recently introduced the second omega index to identify a variety of topological and combinatorial aspects of a graph with a new viewpoint. As a continues study of second omega index, by considering non-incidency of edges, in this paper we introduce the *second omega coindex* $\overline{\Omega_2}$ and then compute it over some well-known graph classes.

2020 MSC: 05C05, 05C50, 05C75

KEYWORDS: Graph, Topologic index, Degree sequence

Introduction

Consider a simple graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E = E(G)$, where $|V(G)| = n$ is the number of vertices and $|E(G)| = m$ is the number of edges. In graph theory, a number that is invariant under graph automorphisms is referred to as a graphical invariant. It is often regarded as a structural invariant relevant to a graph. The term topological index is often reserved for graphical invariant in molecular graph theory. In the mathematical and chemical literature, several dozens of vertex-degree-based graph invariants (usually referred to as "topological indices") have been introduced and extensively studied [1, 3].

One of the newest graph invariant is Second Omega Index, first introduced in [4]. Firstly we recall the definition of second omega invariant:

$$\Omega_2(G) = \sum_{ij\in E} [(i-2)a_i][(j-2)a_j], \tag{1}$$

where $1 \le i, j \le \Delta$. The importance of the second omega index is coming from the consideration of both a graph and a degree sequence of that graph, and so it should be thought as a upgraded version of the omega index [2].

In [4], the second omega indices of some well-known graph classes were calculated. Moreover, this topological index was considered for any tree and the obtained results related to it. The second omega index was given for the derived graphs, including line subdivision and vertex semitotal graph. Besides, the second omega indices of these derived graphs were demonstrated by calculating it for some graph classes.

Main results

In light of the second zagreb coindex and in a quite similar manner, related to nonadjacent edges, we have considered the second omega coindex. After all, for a realizable degree sequence $DS(G) = \{1^{(a_1)}, 2^{(a_2)}, ^{(a_3)}, \ldots, \Delta^{(a_\Delta)}\}$ with its realization G, the second omega coindex is defined as

$$\overline{\Omega_2}(G) = \sum_{ij\notin E} [(i-2)a_i][(j-2)a_j]. \tag{2}$$

The first result will be about the star graphs and direct outcomes of it.

Theorem 1. *For the star graph S_n, we get*

$$\overline{\Omega_2}(S_n) = \frac{(n-2)(n-1)^3}{2}.$$

Proof. Considering the non-adjacent edges of the star graph;

$$\begin{aligned}\overline{\Omega_2}(S_n) &= [1+2+3+...+(n-2)]\left[(1-2)(n-1)^2\right] \\ &= \frac{(n-2)(n-1)^3}{2}.\end{aligned}$$

□

Theorem 2. *For the complete bipartite graph $K_{r,s}$,*

$$\overline{\Omega_2}(K_{r,s}) = \frac{r^3(r-1)(s-2)+s^3(s-1)(r-2)}{2}.$$

Theorem 3. *For the tadpole graph $T_{r,s}$, we get*

$$\overline{\Omega_2}(T_{r,s}) = -1$$

for all $r > 2$.

Theorem 4. *For the wheel graph W_n, the second omega coindex is*

$$\overline{\Omega_2}(W_n) = (\frac{(n-4)(n-3)+2(n-4)}{2})(n-1)^2.$$

Theorem 5. *For the path graph $T_{r,s}$, we get*

$$\overline{\Omega_2}(T_{r,s}) = -1$$

for all $r > 2$.

Theorem 6. *For a path graph P_n, we get*

$$\overline{\Omega_2}(P_n) = 4.$$

Finaly, from above theorems and the results in [4],we arrive at the following Corollary:

Corollary 7. *For the star graph S_n, tadpole graph $T_{r,s}$, path graph P_n and complete graph C_n,*

$$\begin{aligned}&\Omega_2(S_n) \leq \overline{\Omega_2}(S_n) \\ &\overline{\Omega_2}(T_{r,s}) \leq \Omega_2(T_{r,s}) \\ &\Omega_2(P_n) \leq \overline{\Omega_2}(P_n) \\ &\Omega_2(C_n) = \overline{\Omega_2}(C_n) = 0.\end{aligned}$$

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References

[1] J. A. Bondy and U. S. Murty, *Graph theory*, Springer-Verlag, London 2008.

[2] S. Delen and I. N. Cangul, *A new graph invariant*, Turkish Journal of Analysis and Number Theory **6 (1)**, 30–33, 2018.

[3] F. Harary, *Graph theory*, Addison-Wesley, Reading Mass, 1972.

[4] N. U. Ozalan, A. S. Cevik and I. N. Cangul, *The second omega index*, Current Organic Synthesis **21**, 2023; DOI: 10.2174/0115701794250566231115075551.

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Controllability analysis of linear state-delay fractional systems

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In this research paper, our focus is on investigating the controllability of linear time delay differential equations. It is important to differentiate between the notions of function controllability and controllability in Euclidean space (relative controllability) for these equations. This distinction arises because although the solutions of these equations are trajectories in Euclidean space, the natural "state space" is actually a function space. For the purposes of this study, we limit our discussion to controllability in Euclidean space. Furthermore, unlike in the case of ordinary differential equations, it is necessary to also distinguish between the concepts of complete controllability and null controllability when it comes to controllability in Euclidean space.

Chyung and Lee initially explored the concept of complete controllability in Euclidean space, focusing on a linear controlled hereditary system described by multi-delay differential equations [7]. In 1967, Kirillova and Curakova [14] introduced algebraic criteria for the null controllability of linear autonomous time-delay differential equations in Euclidean space. Building upon this work, Gabasov and Curakova [10] demonstrated that the conditions derived in [14] are not only necessary but also sufficient for achieving complete controllability, see also [11], [16]. Weiss [17] extended the understanding of controllability by obtaining an algebraic sufficient condition for time-varying differential-difference equations, encompassing the findings of Buckalo [5] as a special case. Recently, Choudhury [6] published results closely related to those presented by Gabasov and Curakova [10].

In recent decades, the field of fractional calculus has experienced significant advancements due to its broad range of applications in various scientific and engineering domains. Mathematical tools derived from fractional calculus have proven to be highly effective in describing numerous real-world phenomena. These applications encompass diverse areas such as fluid dynamics, arkeology, electrode-electrolyte polarization, transmission modeling, control theory of continuous/discrete dynamical systems, electrical networks, optics, signal processing, and more.

The controllability analysis for fractional linear delay systems is typically based on fractional calculus and control theory. Fractional calculus extends the concept of derivatives and integrals to non-integer orders, allowing the modeling and analysis of systems with fractional dynamics. Fractional delay systems introduce additional complexity due to the presence of fractional orders in the system's dynamics.

The controllability of fractional linear delay systems depends on various factors, including the system's structure, the fractional orders of the delays, and the available control inputs. Fractional order delays can lead to rich and intricate dynamics, and analyzing controllability in such systems can be challenging. Techniques such as fractional differential equations, fractional Laplace transforms, and fractional control theory are commonly used to analyze the controllability properties of fractional linear delay systems.

It is important to note that the field of fractional calculus and fractional control theory is still an active area of research. Developing efficient analysis techniques and control strategies for fractional linear delay systems is an ongoing topic of investigation, and different approaches may be employed depending on the specific system characteristics and requirements.

We study the relative controllability of a linear fractional system with delay

$$\begin{cases} D_{0+}^{\alpha} y(t) = Ay(t) + By(t-h) + Cu(t), \quad t \in (0,T], \ h > 0, \\ \qquad y(0) = y_0, \quad y(t) = \varphi(t), \quad -h \le t < 0. \end{cases} \tag{1}$$

Here $\frac{1}{2} < \alpha \le 1$, A, B are $d \times d$ constant matrices, C is an $d \times r$ constant matrix. We assume that the initial condition $\varphi(t)$ is continuous on the interval $[-h, 0]$ and an admissible control $u \in L^2([0,T], \mathbb{R}^r)$.

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Keywords: Euclidean space, Laplace transform

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References

[1] K. Balachandran and J. Kokila *On the controllability of fractional dynamical systems*, Internat. J. Appl. Math. Comput. Sci. **22**, 523–531, 2012.

[2] K. Balachandran, J. Kokila and J. J. Trujillo, *Relative controllability of fractional dynamical systems with multiple delays in control*, Comput. Math. Appl. **64**, 3037–3045, 2012.

[3] K. Balachandran, J. Y. Park and J. J. Trujillo, *Controllability of nonlinear fractional dynamical systems*, Nonlinear Anal. **75**, 1919–1926, 2012.

[4] D. Baleanu, J. A. T. Machado and A. C. J. Luo, *Fractional dynamics and control*, Springer Science, New York, 2012.

[5] A. Buckalo, *Explicit conditions for controllability of linear systems with time lag*, IEEE Trans. Automatic Control **AC-2**, 193–195, 1968.

[6] A. K. Choudhury, *Algebraic and transfer-function criteria of fixed-time controllability of delay differential systems*, Internat. J. Control **6**, 1073–1082, 1972.

[7] D. H. Chyung and E. B. Lee, *Linear optimal systems with time delay*, SIAM J. Control **4**, 548–574, 1966.

[8] A. Debbouche and D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems*, Comput. Math. Appl. **62**, 1442–1450, 2011.

[9] M. Feckan, J.-R. Wang and Y. Zhou, *Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators*, J. Optim. Theory Appl. **156**, 79–95, 2013.

[10] R. Gabasov and S. V. Churakova, *The theory of controllability of linear systems with delay lags*, Engrg. Cybernetics **4**, 16–27, 1969.

[11] R. Gabasov and F. Kirillova, *The qualitative theory of optimal processes*, Translated from the Russian by John L. Casti Control Systems Theory (Volume 3), Marcel Dekker, Inc., New York-Basel, 1976.

[12] T. L. Guo, *Controllability and observability of impulsive fractional linear time-invariant system*, Comput. Math. Appl. **64**, 3171–3182, 2012.

[13] D. Y. Khusainov and G. V. Shuklin, *Relative controllability in systems with pure delay*, Int J Appl Math. **2**, 210–221, 2005.

[14] F. M. Kirillova and S. V. Churakova, *On the problem of controllability of linear systems with after effects*, Differensial'nye Uravnenija **3**, 436–445, 1967.

[15] M. Li, A. Debbouche and J. R. Wang, *Relative controllability in fractional differential equations with pure delay*, Math. Methods Appl. Sci. **41 (18)** 8906–8914, 2018; DOI: 10.1002/mma.4651.

[16] A. Manitus, *Optimal control o f hereditary systems*, Control Theory and Topics in Functional Analysis **3**, 43–179, 1976.

[17] L. Weiss, *On the controllability of delay-differential equations*, SIAM Journal on Control **5 (4)**, 575–587, 1967; DOI:10.1137/0305036.

[18] R. B. Zmood, *The Euclidean space controllability of control systems with delay*, SIAM J. Control **12 (4)**, 609–623, 1974.

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Estimation of concrete strength by using maturity method

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The strength of concrete is an important mechanical parameter that affects all other properties. Nowadays, different methods are used to estimate the strength of concrete. These are generally divided into two, namely, destructive and non-destructive methods. In addition, maturity relations can be used to obtain information about the current strength of concrete. In the literature, it is seen that the maturity is defined in a temperature-time related manner. This is because strength gain depends on cement hydration, which is a chemical reaction. The development of cement hydration is also related to the temperature at which the concrete is cured and for how long. However, there are other parameters that affect cement hydration, which plays the leading role in determining the strength of concrete.

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KEYWORDS: Strength, Maturity, Hydration

Introduction

Different methods are used to determine the strength of concrete. The maturity calculation is often used to evaluate the strength development of in-situ concrete. The maturity method has been widely used to predict early age concrete strength. However, conventional maturity models exhibit limited predictive ability for late concrete strength under thermal curing conditions due to the influence of the "crossover effect" [7]. When the concept of maturity first flourished, it has been shown that if the temperature gradient of the concrete after the mixing period does not exceed a certain value, the concrete gains strength according to its "maturity" during and after treatment, approximately in accordance with the same law that applies to normally cured concrete [4]. The maturity method as originally proposed is still available today. However, given the aforementioned situation, the maturity calculation needs to be revised. Different scientists have conducted studies on this subject [2, 6, 8]. Quality and safety assurance of concrete structures requires on-site monitoring of early-stage compressive strength development. This is a requirement for all stakeholders involved.

Main results

Saul [4] proposed a single factor, "maturity", as an indicator of concrete strength independent of the combination of temperature and time that constitutes maturity:

$$M = \sum_{t} (T - T_o)\Delta t \tag{1}$$

where M is the maturity ($^\circ C$.day), T is the average temperature over the time interval Δt ($^\circ C$) (20 $^\circ C$ for standard curing), To is the reference temperature ($^\circ C$), Δt is the time interval (days).

One of the most common equations for the strength-maturity relationship is the following logarithmic equation proposed by Plowman [3].

$$S = a + b\ln(M) \tag{2}$$

where S is the compressive strength and a and b are coefficients specific to the concrete mixture.

It is possible to find studies in the literature that test the applicability of the above relations in different material types and curing regimes and/or modify these relations to be used in the strength development of concrete by considering some factors [1, 5].

Conclusion

Concrete is the most widely used material in the construction industry. Therefore, it is important to determine the behavior of concrete used in structures. The most important property of concrete that can be correlated with other properties is compressive strength. Compressive strength also plays a role in the physical feauteures and durability of concrete. In addition, knowing the early age compressive strength development allows the timing of activities such as demolding to be adjusted. Maturity correlations for the strength development of concrete have been proposed in the past and continue to be proposed. In this study, when the parameters affecting the strength development of concrete are taken into account, it is understood that the basic maturity relation may be inadequate, and in such cases it is necessary to modify this relation. However, the equations derived in studies are expected to meet the compressive strength of all concrete types.

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References

[1] W. Liao, B. J. Lee and C. W. Kang, *A humidity-adjusted maturity function for the early age strength prediction of concrete*, Cement and Concrete Composites **30 (6)**, 2008.

[2] D. Miller, N.-M. Ho and N. Talebian, *Monitoring of in-place strength in concrete structures using maturity method–An overview*, Structures **44**, 2022.

[3] J. M. Plowman, *Maturity and the strength of concrete*, Mag Concr Res. **8 (22)**, 1956.

[4] A. G. A. Saul, *Principles underlying the steam curing of concrete at atmospheric pressure*, **2 (6)**, 1951; https://doi.org/10.1680/macr.1951.2.6.127.

[5] Y. Sun and H. S. Lee, *Determination of coefficient "s" and apparent activation energy for fib model code's maturity-based strength function when applied to fly ash concrete*, Construction and Building Materials **409**, 2023.

[6] C. Vázquez-Herrero, I. Martínez-Lage and F. Sánchez-Tembleque, *A new procedure to ensure structural safety based on the maturity method and limit state theory*, Constr. Build. Mater. **35**, 2012.

[7] L. Wang, H. Zhou, J. Zhang, Z. Wang, L. Zhang and M. L. Nehdi, *Prediction of concrete strength considering thermal damage using a modified strength-maturity model*, Construction and Building Materials **400**, 2023; https://doi.org/10.1016/j.conbuildmat.2023.132779.

[8] T. A. Yikici and H.-L.-R. Chen, *Use of maturity method to estimate compressive strength of mass concrete*, Constr. Build. Mater. **95**, 2015.

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A novel formulation of α-simplex and α-MacDonald linear codes with practical applications

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This research contributes to the understanding of linear α-simplex and α-MacDonald Codes, their representations through Gray images, and their applications in multi-secret sharing schemes, thereby advancing the theoretical foundations and practical implementations in the fields of coding theory and information security.

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KEYWORDS: Simplex codes, MacDonald codes, Lineair codes, Gray map, Multi-secret Sharing schemes

Introduction

In the realm of coding theory and information security, the investigation of Linear α-Simplex and α-MacDonald Codes over the ring $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$ has emerged as a compelling area of study. This article delves into three interconnected and crucial facets within this domain, each contributing to our understanding and application of these codes.

The first part explores the mathematical foundations and properties of linear α-Simplex and α-MacDonald Codes when defined over the ring $\mathfrak{R}$. Unveiling the intricacies of these codes within the specified algebraic structure is fundamental to comprehending their potential applications in coding theory.

The second part of this article explores the practical applications of Gray images of linear α-Simplex and α-MacDonald codes in the realm of multi-secret sharing schemes. Investigating how these codes can be effectively employed to enhance the security and robustness of multi-secret sharing systems contributes to the broader field of information security.

As we navigate through these interconnected sections, the goal is to provide a comprehensive exploration of linear α-Simplex and α-MacDonald codes, ranging from their theoretical underpinnings to practical applications in multi-secret sharing. This multifaceted approach aims to contribute to the evolving landscape of coding theory and secure information transmission, emphasizing the versatility and significance of these codes in diverse contexts.

Preliminaries

A finite ring $\mathfrak{R}$ is defined by the product of three commutative rings

$$\mathfrak{R} = \left(\mathbb{Z}_q + v_1\mathbb{Z}_q\right)\left(\mathbb{Z}_q + v_1\mathbb{Z}_q + v_2\mathbb{Z}_q\right)\left(\mathbb{Z}_q + v_1\mathbb{Z}_q + v_2\mathbb{Z}_q + v_3\mathbb{Z}_q\right). \tag{1}$$

Each element in this ring is expressed as $c = (c_1|c_2|c_3)$, where $c_1 \in \mathcal{R}_1$, $c_2 \in \mathcal{R}_2$, and $c_3 \in \mathcal{R}_3$. According to (*cf.* [5, 6]), the components c_1, c_2, and c_3 are represented by

$$c_1 = \left(\frac{1-v_1}{2}\right) c_1^0 + \left(\frac{1+v_1}{2}\right) c_1^1, \tag{2}$$

$$c_2 = \left(1 - \frac{v_1}{\xi_1}\right)\left(1 - \frac{v_2}{\xi_2}\right) c_2^0 + \frac{v_1}{\xi_1}\left(c_2^0 + \xi_1 c_2^1\right) + \frac{v_2}{\xi_2}\left(c_2^0 + \xi_2 c_2^2\right), \tag{3}$$

$$c_3 = \prod_{i=1}^{3}\left(1 - \frac{v_i}{\xi_i}\right) c_3^0 + \frac{v_1}{\xi_1}\left(c_3^0 + \xi_1 c_3^1\right) + \frac{v_2}{\xi_2}\left(c_3^0 + \xi_2 c_3^2\right) + \frac{v_3}{\xi_3}\left(c_3^0 + \xi_3 c_3^3\right). \tag{4}$$

The elements $\mu_0 = \dfrac{1-v_1}{2}$, $\mu_1 = \dfrac{1+v_1}{2}$, $\nu_0^2 = \left(1 - \frac{v_1}{\xi_1}\right)\left(1 - \frac{v_2}{\xi_2}\right)$, $\nu_0^3 = \prod_{i=1}^{3}\left(1 - \dfrac{v_i}{\xi_i}\right)$, $\nu_1^2 = \nu_1^3 = \dfrac{v_1}{\xi_1}$, $\nu_2^2 = \nu_2^3 = \dfrac{v_2}{\xi_2}$ and $\nu_3^3 = \dfrac{v_3}{\xi_3}$ form a fundamental system of idempotents of $\mathcal{R}_1$, $\mathcal{R}_2$ *and* $\mathcal{R}_3$ respectively.

We introduce the Gray map and Gray images for a linear code over the ring $\mathfrak{R}$ into $\mathbb{Z}_q^{72}$ by defining the Gray map for each component ring. This Gray map, denoted as Ψ, is defined as follows:

$$\Psi \quad : \quad \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3 \quad \to \quad \mathbb{Z}_q^{72}.$$

It is evident that this map can straightforwardly extend from $\mathfrak{R}^n$ to $\mathbb{Z}_q^{72n}$. Consequently, the following theorems holds.

Theorem 1. *If C is a linear code over $\mathfrak{R}$ of length n, then $\Phi(C)$ is a linear code with parameters $[72n, k, d_H]$.*

The subsequent theorem provides a characterization of codes and their orthogonality over the ring $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$ with a length of $n = 2n_1 + 3n_2 + 4n_3$.

Theorem 2. *Let C and $C^\perp$ is linear codes of length $n = 2n_1 + 3n_2 + 4n_3$ over $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$, then*

$$C = \left(\bigoplus_{i=0}^{2}\left(\bigoplus_{j=0}^{3} \mu_0 \nu_i^2 \nu_j^3 C_1^0 C_2^i C_3^j\right)\right) \oplus \left(\bigoplus_{i=0}^{2}\left(\bigoplus_{j=0}^{3} \mu_1 \nu_i^2 \nu_j^3 C_1^1 C_2^i C_3^j\right)\right), \tag{5}$$

and

$$C^\perp = \left(\bigoplus_{i=0}^{2}\left(\bigoplus_{j=0}^{3} \mu_0 \nu_i^2 \nu_j^3 C_1^{0\perp} C_2^{i\perp} C_3^{j\perp}\right)\right) \oplus \left(\bigoplus_{i=0}^{2}\left(\bigoplus_{j=0}^{3} \mu_1 \nu_i^2 \nu_j^3 C_1^{1\perp} C_2^{i\perp} C_3^{j\perp}\right)\right).$$

Linear α-simplex and α-MacDonald codes over $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$

In the referenced works (*cf.* [2, 3, 4]), the authors have explored the construction of simplex and MacDonald linear codes of type α characteristics across specific rings. In this study, a novel representation of these codes over the ring $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$ has been established by utilizing the idempotent elements of this ring.

Theorem 3. *Let $m^\alpha_{k,\mathcal{R}_1}$, $G^\alpha_{k,\mathcal{R}_2}$ and $\mathcal{G}^\alpha_{k,\mathcal{R}_3}$ be generator matrices of linear α-simplex codes over $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ respectively. Then, the generator matrix $\Theta^\alpha_{k,\mathfrak{R}}$ of linear α-simplex codes over $\mathfrak{R}$ as follows*

$$\begin{aligned}\Theta^\alpha_{k,\mathfrak{R}} &= q^{26}\mu_0\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3\left[\sigma_0(m^\alpha_{k,\mathcal{R}_1})\right]\left[\lambda_j^i(G^\alpha_{k,\mathcal{R}_2})\right]\left[\gamma_j^i(\mathcal{G}^\alpha_{k,\mathcal{R}_3})\right]\right]\right]\\ &\oplus q^{26}\mu_1\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3\left[\sigma_1(m^\alpha_{k,\mathcal{R}_1})\right]\left[\lambda_j^i(G^\alpha_{k,\mathcal{R}_2})\right]\left[\gamma_j^i(\mathcal{G}^\alpha_{k,\mathcal{R}_3})\right]\right]\right],\end{aligned}$$

such that $\sigma_0(m^\alpha_{k,\mathcal{R}_1})$, $\sigma_1(m^\alpha_{k,\mathcal{R}_1})$, $\lambda_j^i(G^\alpha_{k,\mathcal{R}_2})$ and $\gamma_j^i(\mathcal{G}^\alpha_{k,\mathcal{R}_3})$, for $0 \leq i \leq 2$ and $0 \leq j \leq 3$ are equivalent matrices.

Proposition 4. *A linear α-simplex code S^α_k of length $n = 2n_1 + 3n_2 + 4n_3$ over $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$, can be expressed by*

$$\begin{aligned}S^\alpha_k &= q^{26k}\mu_0\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3\left[\sigma_0(S^\alpha_{k,\mathcal{R}_1})\right]\left[\lambda_j^i(S^\alpha_{k,\mathcal{R}_2})][\gamma_j^i(S^\alpha_{k,\mathcal{R}_3})\right]\right]\right]\\ &\oplus q^{26k}\mu_1\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3[\sigma_1(S^\alpha_{k,\mathcal{R}_1})\left[\lambda_j^i(S^\alpha_{k,\mathcal{R}_2})\right]\left[\gamma_j^i(S^\alpha_{k,\mathcal{R}_3})\right]\right]\right].\end{aligned}$$

Additionally, the α-MacDonald codes over the ring $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$ are presented as follows:

Corollary 5. *Let $m^\alpha_{k,u,\mathcal{R}_1}$, $G^\alpha_{k,u,\mathcal{R}_2}$ and $\mathcal{G}^\alpha_{k,u,\mathcal{R}_3}$ be the generator matrices of α-MacDonald codes over $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ respectively.Then, the generator matrix $\Theta^\alpha_{\mathfrak{R},k,u}$ of α-MacDonald codes over $\mathfrak{R}$ as follows*

$$\begin{aligned}\Theta^\alpha_{k,u,\mathfrak{R}} &= q^{26k}\mu_0\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3\left[\sigma_0(m^\alpha_{k,u,\mathcal{R}_1})\right]\left[\lambda_j^i(G^\alpha_{k,u,\mathcal{R}_2})\right]\left[\gamma_j^i(\mathcal{G}^\alpha_{k,u,\mathcal{R}_3})\right]\right]\right]\\ &\oplus q^{26k}\mu_1\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3\left[\sigma_1(m^\alpha_{k,u,\mathcal{R}_1})\right]\left[\lambda_j^i(G^\alpha_{k,u,\mathcal{R}_2})\right]\left[\gamma_j^i(\mathcal{G}^\alpha_{k,u,\mathcal{R}_3})\right]\right]\right],\end{aligned}$$

with $\sigma_0(m^\alpha_{k,u,\mathcal{R}_1})$, $\sigma_1(m^\alpha_{k,u,\mathcal{R}_1})$, $\lambda_j^i(G^\alpha_{k,u,\mathcal{R}_2})$ and $\gamma_j^i(\mathcal{G}^\alpha_{k,u,\mathcal{R}_3})$, for $0 \leq i \leq 2$ and $0 \leq j \leq 3$ are equivalent matrices.

Corollary 6. *A linear α-MacDonald code $\mathcal{M}^\alpha_k$ of length $n = 2n_1 + 3n_2 + 4n_3$ over $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$, can be expressed by*

$$\begin{aligned}\mathcal{M}^\alpha_k &= q^{26k}\mu_0\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3\left[\sigma_0(\mathcal{M}^\alpha_{k,\mathcal{R}_1})\right]\left[\lambda_j^i(\mathcal{M}^\alpha_{k,\mathcal{R}_2})\right]\left[\gamma_j^i(\mathcal{M}^\alpha_{k,\mathcal{R}_3})\right]\right]\right]\\ &\oplus q^{26}\mu_1\left[\bigoplus_{i=0}^{2}\left[\bigoplus_{j=0}^{3}\nu_i^2\nu_j^3\left[\sigma_1(\mathcal{M}^\alpha_{k,\mathcal{R}_1})\right]\left[\lambda_j^i(\mathcal{M}^\alpha_{k,\mathcal{R}_2})\right]\left[\gamma_j^i(\mathcal{M}^\alpha_{k,\mathcal{R}_3})\right]\right]\right].\end{aligned}$$

Gray images of linear α-simplex and α-MacDonald codes

Within the domain of coding theory, the investigation of gray images gains heightened significance, especially in the context of the linear α-simplex and α-MacDonald codes. These codes embody intricate mathematical structures with versatile applications across various domains, including the realm of multi-secret sharing schemes.

Theorem 7. *Let S_k^α be a linear α-simplex code of length $n = 2n_1 + 3n_2 + 4n_3$ over $\mathfrak{R}$, then*

$$\begin{aligned}\Phi(S_k^\alpha) &= q^{26}\left[\bigotimes_{i=0}^{2}\left[\bigotimes_{j=0}^{3}(\sigma_0(S_{k,\mathbb{Z}_q}^\alpha)\lambda_j^i(S_{k,\mathbb{Z}_q}^\alpha)\gamma_j^i(S_{k,\mathbb{Z}_q}^\alpha)\right]\right]\\ &\bigotimes q^{26}\left[\bigotimes_{i=0}^{2}\left[\bigotimes_{j=0}^{3}\sigma_1(S_{k,\mathbb{Z}_q}^\alpha)\lambda_j^i(S_{k,\mathbb{Z}_q}^\alpha)\gamma_j^i(S_{k,\mathbb{Z}_q}^\alpha)\right]\right],\end{aligned}$$

is $[72n, k, d]$-linear α-simplex codes over $\mathbb{Z}_q$.

Corollary 8. *The generator matrix of $\Phi(\Theta_{k,\mathfrak{R}}^\alpha)$ is a permutation equivalent of the matrix*

$$\Phi(\Theta_{k,\mathfrak{R}}^\alpha) = \left[\overbrace{M_{k,\mathbb{Z}_q}^\alpha M_{k,\mathbb{Z}_q}^\alpha \dots M_{k,\mathbb{Z}_q}^\alpha}^{q^{144k}}\right],$$

with $M_{k,\mathbb{Z}_q}^\alpha$ is a generator matrix of $S_{k,\mathbb{Z}_q}^\alpha$ over $\mathbb{Z}_q$.

Theorem 9. *Let $\mathcal{M}_k^\alpha$ be a linear α-MacDonald code of length $n = 2n_1 + 3n_2 + 4n_3$ over $\mathfrak{R}$, then*

$$\begin{aligned}\Phi(\mathcal{M}_k^\alpha) &= q^{26k}\left[\bigotimes_{i=0}^{2}\left[\bigotimes_{j=0}^{3}(\sigma_0(\mathcal{M}_{k,\mathcal{R}_1}^\alpha)\lambda_j^i(\mathcal{M}_{k,\mathcal{R}_2}^\alpha)\gamma_j^i(\mathcal{M}_{k,\mathcal{R}_3}^\alpha)\right]\right]\\ &\bigotimes q^{26k}\left[\bigotimes_{i=0}^{2}\left[\bigotimes_{j=0}^{3}(\sigma_0(\mathcal{M}_{k,\mathcal{R}_1}^\alpha)\lambda_j^i(\mathcal{M}_{k,\mathcal{R}_2}^\alpha)\gamma_j^i(\mathcal{M}_{k,\mathcal{R}_3}^\alpha)\right]\right],\end{aligned}$$

is $[72n, k, d]$ linear α-MacDonald code over $\mathbb{Z}_q$.

Corollary 10. *The generator matrix of $\Phi(\Theta_{k,\mathfrak{R}})$ is a permutation equivalent of the matrix*

$$\Phi(\Theta_{k,u,\mathfrak{R}}^\alpha) = \left[\overbrace{M_{k,u,\mathbb{Z}_q}^\alpha M_{k,u,\mathbb{Z}_q}^\alpha \cdots M_{k,u,\mathbb{Z}_q}^\alpha}^{q^{144k}}\right], \tag{6}$$

with $M_{k,u,\mathbb{Z}_q}^\alpha$ is a generator matrix of $\mathcal{M}_{k,u,\mathbb{Z}_q}^\alpha$ over $\mathbb{Z}_q$.

Multi-secret sharing schemes based on α-Simplex and α-MacDonald codes

- Minimal linear codes constitute a unique category of codes that find notable applications in secret-sharing schemes and multi-secret sharing schemes.
- In our research, we make use of the Gray images of α-MacDonald subcodes over $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$ due to their inherent minimality. This choice leads us to employ multi-secret sharing schemes as a consequence.
- According to (*cf.* [1]), we construct the Multi-secret sharing schemes based on linear subcodes.
- We need an α-MacDonald subcode over $\mathbb{Z}_q$ with a generator matrix $\Phi(\Theta_{k,\mathfrak{R}}^\alpha)$.
- To ensure the uniqueness of the solution to the linear system, it is imperative that the code verifies $C\ is\ LCD\ code\ if\ only\ \ if\ rank(G) = rank(GG^\top) = rank(G^\top G) \neq 0$ for matrix G.
- Let a codeword be the secret $S = (s_1, s_2, \dots, s_n)$ in $\mathbb{Z}_q^n$, where S^T denotes the transposition of S.

- The rows $\{g_1, g_2 \ldots, g_n\}$ of the generator matrix $\Phi(\Theta^{\alpha}_{k,\mathfrak{R}})$ are minimal access elements, and all elements of $\Phi(\Theta^{\alpha}_{k,\mathfrak{R}})$ are participants in this scheme.
- The dealer, knowing the secret S, computes the share u of the user with attached codeword c, by taking the scalar product of that codeword with the secret, i.e.,

$$u = \langle c, s\rangle = c \cdot S^T. \tag{7}$$

- Consider the system with the private secret S and the coalition corresponding to the rows of $\Phi(\Theta^{\alpha}_{k,\mathfrak{R}})$, we have

$$U^T = \Phi(\Theta^{\alpha}_{k,\mathfrak{R}}) \cdot S^T, \tag{8}$$

  where $U = (u_1, u_2, \ldots, u_k)$, and u_i is the share attached to the row i of $\Phi(\Theta^{\alpha}_{k,u,\mathfrak{R}})$.
- The secret can then be calculated by solving the following linear system of n equations and n unknowns:

$$\begin{cases} U^T &= \Phi(\Theta^{\alpha}_{k,\mathfrak{R}}) \cdot S^T \\ 0 &= H(\Theta^{\alpha}_{2,1,\mathfrak{R}}) \cdot S^T. \end{cases} \tag{9}$$

Example 11. *Let* $\mathfrak{R} = (\mathbb{Z}_3 + v_1\mathbb{Z}_3)(\mathbb{Z}_3 + v_1\mathbb{Z}_3 + v_2\mathbb{Z}_3)(\mathbb{Z}_3 + v_1\mathbb{Z}_3 + v_2\mathbb{Z}_3 + v_3\mathbb{Z}_3)$*. For* $k = 2$*, a generator matrix of subcode of* $\Phi\left(\mathcal{M}^{\alpha}_{2,1,\mathfrak{R}}\right)$ *is given by*

$$\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}}) \quad = \quad \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix},$$

and

$$rank(\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}})) = rank(\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}})^{\top}\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}})) = rank(\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}})\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}})^{\top}) = 2,$$

the subcode $\Phi\left(\mathcal{M}'^{\alpha}_{2,1,\mathfrak{R}}\right)$ *is LCD. The parity-check matrix* $H(\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}}))$ *of this subcode is*

$$H(\Phi(\Theta'^{\alpha}_{2,1,\mathfrak{R}})) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{10}$$

There are 9 *codewords of* $\Phi\left(\mathcal{M}'^{\alpha}_{2,1,\mathfrak{R}}\right)$,

$$2001, 0210, 1122, 0120, 1002, 2211, 1212, 2121, 0000.$$

Now, we examine a Multi-secret sharing schemes based on $\Phi\left(\mathcal{M}'^{\alpha}_{2,1,\mathfrak{R}}\right)$*. Let the secret vector be* $S = 2121$*. We calculate the shares as follows*

$$u_1^{\top} =< 1122, 2121 >= 0$$
$$u_2^{\top} =< 1212, 2121 >= 2.$$

Moreover,

$$d_1^{\top} =< 1001, 2121 >= 0$$
$$d_2^{\top} =< 0110, 2121 >= 0.$$

Using (9)*, we should solve the following linear system to recover the secret.*

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}. \tag{11}$$

According to (cf. [7, Theorem 6]), this system has a unique solution. So, we recove the secret $S = 2121$*.*

Conclusion

In conclusion, the exploration of linear α-simplex and α-MacDonald Codes over the algebraic structure $\mathfrak{R} = \mathcal{R}_1\mathcal{R}_2\mathcal{R}_3$ has unveiled a rich landscape of coding theory with promising applications. Furthermore, the development of multi-secret sharing schemes based on α-simplex and α-MacDonald Codes expands the horizons of secure communication protocols.

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References

[1] A. Alahmadi, A. Altassan, A. AlKenani, S. Çalkavur, S. Hatoon and S. Patrick, *A Multi-secret sharing scheme based on LCD codes*, Mathematics **8 (2)**, 2020; Article ID: 272.

[2] K. Chatouh, K. Guenda, T.A. Gulliver and L. Noui, *On some classes of linear codes over* $\mathbb{Z}_2\mathbb{Z}_4$ *and their covering radii*, J. Appl. Math. Comput. **53 (1)**, 201–222, 2017.

[3] K. Chatouh, K. Guenda and T.A. Gulliver, L. Noui, *Simplex and MacDonald codes over* R_q, J. Appl. Math. Comput. **55 (1-2)**, 455–478, 2017.

[4] K. Chatouh, K. Guenda and T.A. Gulliver, *New classes of codes over* $R_{q,p,m} = \mathbb{Z}_{p^m}[u_1, u_2, \cdots, u_q]/\left\langle u_i^2 = 0, u_iu_j = u_ju_i \right\rangle$ *and their applications*, Comput. Appl. Math. **39 (3)**, 1–39, 2020.

[5] I. Habibul, P. Shikha, O.M. Prakash and S. Patrick, *A family of constacyclic codes over a class of non-chain rings* $A_{q,r}$ *and new quantum codes*, J. Appl. Math. Comput. 2021; hal-03324685.

[6] W. Rongsheng and S. Minjia, *On* $\mathbb{Z}_2\mathbb{Z}_4$*-additive polycyclic codes and their Gray images*, Designs, Codes and Cryptography 2021; https://doi.org/10.1007/s10623-021-00917-0.

[7] L. Zihui, *Galois LCD codes over rings*, Adv. Math. Commun. **18 (1)**, 91–104, 2024; DOI: 10.3934/amc.2022002.

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The spaces of multilinear multipliers

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In this paper, we define the multilinear multipliers of the function space $A_{w,\omega}^{p,q}(\mathbb{R})$ that was defined and investigated by R. H. Fischer, A. T. Gürkanli and T. S. Liu. Then we investigate properties of the these multipliers. Furthermore, we give some examples for multilinear multipliers.

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Keywords: Multilinear multipliers, Weighted Lebesgue space

Introduction

Throughout this paper we will work on $\mathbb{R}^n$ with Lebesgue measure dx. We will denote by $C_c(\mathbb{R}^n)$ and $C_c^{\infty}(\mathbb{R}^n)$ the space of all continuous, complex-valued functions with compact support on $\mathbb{R}^n$ and the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}^n$ respectively. Also we will denote by $S(\mathbb{R}^n)$ the Schwartz class of functions. In this work we will use the Beurling weight functions, i.e., real valued, measurable and locally bounded functions w on $\mathbb{R}^n$ which satisfy $w(x) \geq 1$ and $w(x+y) \leq w(x)\,w(y)$ for all $x, y \in \mathbb{R}^d$. For $1 \leq p < \infty$, the spaces $L_w^p(\mathbb{R}^n)$ are the weighted Lebesgue spaces on $\mathbb{R}^n$. It is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$. Moreover, $L_w^1(\mathbb{R}^n)$ is a Banach convolution algebra. It is called a Beurling algebra, [3, 10]. A weight function v_s is of polynomial type if

$$v_s(x) = (1+|x|)^s$$

for $s \geq 0$. A weight function w is said to satisfy the Beurling- Domar condition (shortly BD) if one has

$$\sum_{n=0}^{\infty} \frac{\log w(nx)}{n^2} < \infty, \ \forall x \in \mathbb{R}^n, \tag{1}$$

(*cf.* [2, 9]).

As an example the polynomial type weight function $v_s(x)$ satisfies the BD-condition, [12] . The Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle x,t\rangle} dx,$$

where $\langle x,t\rangle = \sum_{i=1}^{n} x_i t_i$ be the usual scalar product on $\mathbb{R}^n$. It is known that $\hat{f} \in C_0(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ denotes the space of complex-valued continuous functions on $\mathbb{R}^n$ that vanish at infinity. For a Borel measure μ we denote by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi,x\rangle} d\mu(x)$$

its Fourier transform. Also we denote by $M(w)$, the set of Borel measures μ for which

$$\int_{\mathbb{R}^n} wd|\mu|(x) < \infty,$$

(*cf.* [5]).

Let $1 \leq p, q < \infty$ and w, ω be weight functions on $\mathbb{R}^n$. We set

$$A_{w,\omega}^{p,q}(\mathbb{R}^n) = \left\{ f \in L_w^p(\mathbb{R}^n) \mid \hat{f} \in L_w^q(\mathbb{R}^n) \right\} \tag{2}$$

and equip this space with the norm

$$\left\| f \mid A_{w,\omega}^{p,q}(\mathbb{R}^n) \right\| = \|f\|_{p,w} + \left\| \hat{f} \right\|_{q,\omega}, \tag{3}$$

where $(\hat{\cdot})$ is the generalized Fourier transform (*cf.* [4]).

This space was defined and investigated by R. H. Fischer, A. T. Gürkanlı and T. S. Liu [4]. It is known that $A_{w,\omega}^{p,q}(\mathbb{R}^n)$ is a Banach space, if the first weight w satisfies (BD), then $A_{w,\omega}^{p,q}(\mathbb{R}^n)$ admits an approximate identity (shortly AI) bounded in $L_w^1(\mathbb{R}^n)$ and with compactly supported Fourier transforms, and furthermore $A_{w,\omega}^{p,q}(\mathbb{R}^n)$ is an essential Banach module over $L_w^1(\mathbb{R}^n)$, [4]. It is also known by Proposition 1.13 in [4] that if the weight function w on $\mathbb{R}^n$ satisfies the condition (BD), then the set

$$\Lambda_w^p = \left\{ f \in L_w^p(\mathbb{R}^n) \mid \text{supp}\hat{f} \text{ is compact} \right\} \tag{4}$$

is dense in $A_{w,\omega}^{p,q}(\mathbb{R}^n)$.

The bounded function $m(\xi, \eta)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be bilinear multiplier on $\mathbb{R}^n$ of type (p_1, p_2, p_3) if

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta, \tag{5}$$

defines a bounded bilinear operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$. The study of bilinear multipliers goes back to the work Coifman and Meyer in [1]. The some results were extended by Grafakos and Torres [6]. Kulak and Gürkanlı extended these results to weighted Lebesgue and variable exponent Lebesgue spaces [8].

Main results

In this Section we define the bilinear multipliers from $A_{w_1,\omega_1}^{p_1,q_1}(\mathbb{R}) \times \cdots \times A_{w_d,\omega_d}^{p_d,q_d}(\mathbb{R})$ into $A_{w_{d+1},\omega_{d+1}}^{p_{d+1},q_{d+1}}(\mathbb{R})$, and investigate some properties of the space of these multilinear multipliers. Throughout this work we will assume that w_i $(i = 1, \ldots, d)$ satisfy Beurling Domar (BD) condition.

Definition 1. *Let $1 \leq p_i, q_i, r_i < \infty$ and w_i, ω_i $(i = 1, \ldots, d+1)$ be weight functions on $\mathbb{R}$. Assume that $m(\xi_1, \ldots, \xi_d)$ is locally integrable function on $\mathbb{R}^d$. Define*

$$B_m(f_1, \ldots, f_d)(x) = \int_{\mathbb{R}^d} \widehat{f_1}(\xi_1), \ldots, \widehat{f_d}(\xi_d) m(\xi_1, \ldots, \xi_d) e^{2\pi i \langle \xi_1 + \cdots + \xi_d, x \rangle} d\xi_1 \ldots d\xi_d \tag{6}$$

for all $f_i \in \Lambda_{w_i}^{p_i}$, $(i = 1, \ldots, d)$. m is said to be bilinear multiplier on $\mathbb{R}$ of type $A(F)(p_i, q_i, w_i, v_i)$ (shortly $A(F)$), if there exists $C > 0$ such that

$$\|B_m(f_1, \ldots, f_d)\|_{A_{w_{d+1},\omega_{d+1}}^{p_{d+1},q_{d+1}}} \leq C \prod_{i=1}^{d} \|f_i\|_{A_{w_i,\omega_i}^{p_i,q_i}}$$

for all $f_i \in \Lambda^{p_i}_{w_i}$, $(i = 1, \ldots, d)$. *That means* B_m *extends to a bounded multilinear operator from* $A^{p_1,q_1}_{w_1,\omega_1}(\mathbb{R}) \times \cdots \times A^{p_d,q_d}_{w_d,\omega_d}(\mathbb{R})$ *into* $A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}(\mathbb{R})$. $BM\left[A(F)(p_i, q_i, w_i, v_i)\right]$ *(shortly* $BM\left[A(F)\right]$*) denotes the space of all multilinear multipliers of type* $A(F)(p_i, q_i, w_i, v_i)$. *We denote by*

$$\begin{aligned}\|m\|_{A(F)} &= \|B_m\| \\ &= \sup\left\{ \frac{\|B_m(f_1, \ldots, f_d)\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}}}{\prod_{i=1}^{d} \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}}} : \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} \leq 1 \ (i = 1, \ldots, d) \right\}.\end{aligned}$$

Lemma 2 (**Hölder Type Inequality**). *Assume that* $w < \prod_{i=1}^{d} w_i$ *and* $\omega < \prod_{i=1}^{d} \omega_i$. *If one has the equalities* $\frac{1}{p_1} + \cdots + \frac{1}{p_d} = \frac{1}{p}$ *and* $\frac{1}{q_1} + \cdots + \frac{1}{q_d} - (d-1) = \frac{1}{q}$, *then the following inequality holds*

$$\left\| \prod_{i=1}^{d} f_i \right\|_{A^{p,q}_{w,\omega}} \leq \prod_{i=1}^{d} \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}},$$

where $f_i \in A^{p_i,q_i}_{w_i,\omega_i}(\mathbb{R})$, $(i = 1, \ldots, d)$.

Proof. Let $f_i \in A^{p_i,q_i}_{w_i,\omega_i}(\mathbb{R})$ $(i = 1, \ldots, d)$. It's known that the equality

$$\left(\prod_{i=1}^{d} f_i\right)^{\hat{}} = \hat{f}_1 * \cdots * \hat{f}_d.$$

From the this equality, generalized Hölder inequaltiy and Young inequality, we have

$$\begin{aligned}\left\| \prod_{i=1}^{d} f_i \right\|_{A^{p,q}_{w,\omega}} &= \left\| \prod_{i=1}^{d} f_i \right\|_{p,w} + \left\| \left(\prod_{i=1}^{d} f_i\right)^{\hat{}} \right\|_{q,\omega} \\ &\leq \prod_{i=1}^{d} \|f_i\|_{p,w} + \left\| \hat{f}_1 * \cdots * \hat{f}_d \right\|_{q,\omega} \\ &\leq \prod_{i=1}^{d} \|f_i\|_{p_i,w} + \prod_{i=1}^{d} \left\| \hat{f}_i \right\|_{q_i,\omega}.\end{aligned}$$

$$\prod_{i=1}^{d} \|f_i\|_{p_i,w_i} + \prod_{i=1}^{d} \left\| \hat{f}_i \right\|_{q_i,\omega_i} \leq \prod_{i=1}^{d} \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}}.$$

□

Proposition 3. *Let* $\frac{1}{p_1} + \cdots + \frac{1}{p_d} = \frac{1}{p_{d+1}}$ *and* $\frac{1}{q_1} + \cdots + \frac{1}{q_d} - (d-1) = \frac{1}{q_{d+1}}$, *and let* w_i $(i = 2, \ldots, d)$ *be symmetric weight functions on* $\mathbb{R}$. *Assume that* $w_{d+1} < \prod_{i=1}^{d} w_i$ *and* $\omega_{d+1} < \prod_{i=1}^{d} \omega_i$ *and* $w = \prod_{i=1}^{d} w_i$. *If* $K \in L^1_w(\mathbb{R})$, *then* $m(\xi_1, \ldots, \xi_d) = \hat{K}(\xi_1 - \xi_2 - \cdots - \xi_d)$ *defines a multilinear multiplier and*

$$\|m\|_{A(F)} \leq \|K\|_{1,w}.$$

Proof. For $f_i \in \Lambda^{p_i}_{w_i}$, $(i = 1, \ldots, d)$, we know that

$$B_m(f_1, \ldots, f_d)(x) = \int\limits_{\mathbb{R}} f_1(x-y) f_2(x+y) \ldots f_d(x+y) K(y) dy \ . \tag{7}$$

Then by (7)

$$\begin{aligned}
\|B_m(f_1,\ldots,f_d)\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} &\le \int\limits_{\mathbb{R}} \|T_y f_1 T_{-y} f_2 \ldots T_{-y} f_d K(y)\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} dy \\
&= \int\limits_{\mathbb{R}} \|T_y f_1 T_{-y} f_2 \ldots T_{-y} f_d\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} |K(y)|\, dy \\
&= \int\limits_{\mathbb{R}} \left\| T_y f_1 \prod_{i=2}^{d} T_{-y} f_i \right\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} |K(y)|\, dy. \qquad (8)
\end{aligned}$$

Using Lemma 2, the inequality (8) and the hypothesis we have

$$\|B_m(f_1,\ldots,f_d)\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} \le \int\limits_{\mathbb{R}} \|T_y f_1\|_{A^{p_1,q_1}_{w_1,\omega_1}} \prod_{i=2}^{d} \|T_{-y} f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} |K(y)|\, dy$$

$$\begin{aligned}
&= \int\limits_{\mathbb{R}} \left(\|T_y f_1\|_{p_i,w_1} + \left\| (T_y f_1)\hat{}\, \right\|_{q_i,\omega_1} \right) \\
&\quad \times \prod_{i=2}^{d} \left(\|T_{-y} f_i\|_{p_i,w_i} + \left\| (T_{-y} f_i)\hat{}\, \right\|_{q_i,\omega_i} \right) |K(y)|\, dy \\
&= \int\limits_{\mathbb{R}} \left(w_1(y) \|f_1\|_{p_i,w_1} + \left\| (T_y f_1)\hat{}\, \right\|_{q_i,\omega_1} \right) \\
&\quad \times \prod_{i=2}^{d} \left(w_i(-y) \|f_i\|_{p_i,w_i} + \left\| (T_{-y} f_i)\hat{}\, \right\|_{q_i,\omega_i} \right) |K(y)|\, dy \\
&= \int\limits_{\mathbb{R}} \left(w_1(y) \|f_1\|_{p_i,w_1} + \left\| M_y \hat{f}_1 \right\|_{q_i,\omega_1} \right) \\
&\quad \times \prod_{i=2}^{d} \left(w_i(-y) \|f_i\|_{p_i,w} + \left\| M_{-y} \hat{f}_i \right\|_{q_i,\omega_i} \right) |K(y)|\, dy \\
&= \int\limits_{\mathbb{R}} \left(w_1(y) \|f_1\|_{p_i,w_1} + \left\| \hat{f}_1 \right\|_{q_i,\omega_1} \right) \\
&\quad \times \prod_{i=2}^{d} \left(w_i(y) \|f_i\|_{p_i,w} + \left\| \hat{f}_i \right\|_{q_i,\omega_i} \right) |K(y)|\, dy \\
&\le \int\limits_{\mathbb{R}} \left(w_1(y) \|f_1\|_{p_i,w_1} + w_1(y) \left\| \hat{f}_1 \right\|_{q_i,\omega_1} \right) \\
&\quad \times \prod_{i=2}^{d} \left(w_i(y) \|f_i\|_{p_i,w} + w_i(y) \left\| \hat{f}_i \right\|_{q_i,\omega_i} \right) |K(y)|\, dy \\
&= \prod_{i=1}^{d} \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} \int\limits_{\mathbb{R}} |K(y)| \prod_{i=1}^{d} w_i(y)\, dy = \prod_{i=1}^{d} \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} \|K\|_{1,w} \, . \qquad (9)
\end{aligned}$$

Finally by (9), we obtain

$$\|m\|_{A(F)} \leq \|K\|_{1,w}\,.$$

This completes the proof. $\square$

This theorem motivates us restrict ourselves to the following special classes of multilinear multipliers.

Definition 4. *Let $1 \leq p_i, q_i < \infty$ and ω_i, v_i $(i = 1, \ldots, d+1)$ be weight functions on $\mathbb{R}$. We denote by $\tilde{M}\ [A\,(F)\,(p_i, q_i, w_i, \omega_i)]$ the space of measurable functions $M : \mathbb{R} \to \mathbb{C}$ such that $m(\xi_1, \ldots, \xi_d) = M(\xi_1 - \xi_2 - \cdots - \xi_d) \in \tilde{M}\,[A\,(F)\,(p_i, q_i, w_i, \omega_i)]$, that is to say*

$$B_m(f_1, \ldots, f_d)(x) = \int\limits_{\mathbb{R}^d} \hat{f}_1(\xi_1) \ldots \hat{f}_d(\xi_d) M(\xi_1 - \xi_2 - \cdots - \xi_d) e^{2\pi i \langle \xi_1 + \cdots + \xi_d \rangle} d\xi_1 \ldots d\xi_d$$

extends to a bounded bilinear map from $A^{p_1,q_1}_{w_1,\omega_1}(\mathbb{R}) \times \cdots \times A^{p_d,q_d}_{w_d,\omega_d}(\mathbb{R})$ to $A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}(\mathbb{R})$. We denote $\|M\|_{A(F)} = \|B_M\|$.

Now we will give an example of multilinear multiplier.

Proposition 5. *Let $\frac{1}{p_1} + \cdots + \frac{1}{p_d} = \frac{1}{p_{d+1}}$, $\frac{1}{q_1} + \cdots + \frac{1}{q_d} - (d-1) = \frac{1}{q_{d+1}}$, and let $w_{d+1} < \prod\limits_{i=1}^{d} w_i$, $\omega_{d+1} < \prod\limits_{i=1}^{d} \omega_i$. Assume that $\mu \in M\,(v)$ and $w_i\,(\alpha x) \leq |\alpha|\, w_i\,(x)$, $(i = 1, \ldots, d)$ for all $\alpha \in \mathbb{R}$. Then $m(\xi_1, \ldots, \xi_d) = \hat{\mu}(\alpha_1 \xi_1 + \cdots + \alpha_d \xi_d)$ for $\alpha_i \in \mathbb{R}$ $(i = 1, \ldots, d)$ defines a multilinear multiplier and*

$$\|m\|_{A(F)} \leq C\, \|\mu\|_{M(v)}\,,$$

where $C = \prod\limits_{i=1}^{d} |\alpha_i|$.

Proof. First we write the following equality by

$$B_m(f_1, \ldots, f_d)(x) = \int\limits_{\mathbb{R}} f_1(x - \alpha_1 t) \ldots f_d(x - \alpha_d t) d\mu(t), \tag{10}$$

for any $f_i \in \Lambda^{p_i}_{w_i}$ $(i = 1, \ldots, d)$. By using (10) and Lemma 2

$$\|B_m(f_1, \ldots, f_d)\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} = \left\| \int\limits_{\mathbb{R}} f_1(x - \alpha_1 t) \ldots f_d(x - \alpha_d t) d\mu(t) \right\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}}$$

$$\leq \int\limits_{\mathbb{R}} \|f_1(x - \alpha_1 t) \ldots f_d(x - \alpha_d t)\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} d\,|\mu|\,(t)\,.$$

$$\int\limits_{\mathbb{R}} \prod_{i=1}^{d} \|T_{\alpha_i t} f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} d\,|\mu|\,(t)\,. \tag{11}$$

Then we have

$$\begin{aligned}\|T_{\alpha_i t} f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} &= \|T_{\alpha_i t} f_i\|_{p_i,w_i} + \left\|(T_{\alpha_i t} f_i)^{\hat{}}\right\|_{q_i,\omega_i} \\ &\leq w_i(\alpha_i t)\|f_i\|_{p_i,w_i} + w_i(\alpha_i t)\left\|M_{\alpha_i t}\hat{f}_i\right\|_{q_i,\omega_i} \\ &\leq |\alpha_i| w_i(t)\left(\|f_i\|_{p_i,w_i} + \left\|\hat{f}_i\right\|_{q_i,\omega_i}\right) = |\alpha_i| w_i(t)\|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}}\end{aligned}$$

for $i = 1, \ldots, d$. So combining the inequalities (10) and (11),

$$\|B_m(f_1,\ldots,f_d)\|_{A^{p_{d+1},q_{d+1}}_{w_{d+1},\omega_{d+1}}} \leq \int_{\mathbb{R}} \prod_{i=1}^{d} |\alpha_i| w_i(t)\|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} d|\mu|(t)$$

$$= \prod_{i=1}^{d} |\alpha_i| \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} \int_{\mathbb{R}} \prod_{i=1}^{d} w_i(t)\, d|\mu|(t) = \prod_{i=1}^{d} |\alpha_i| \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} \int_{\mathbb{R}} v(t)\, d|\mu|(t)$$

$$= \prod_{i=1}^{d} |\alpha_i| \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} \int_{\mathbb{R}} v(t)\, d|\mu|(t) = \prod_{i=1}^{d} |\alpha_i| \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}} \|\mu\|_{M(v)}$$

$$= C\|\mu\|_{M(v)} \prod_{i=1}^{d} \|f_i\|_{A^{p_i,q_i}_{w_i,\omega_i}},$$

where $C = \prod_{i=1}^{d} |\alpha_i|$. This implies

$$\|B_m\| = \|m\|_{A(F)} \leq C\|\mu\|_{M(v)}.$$

□

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References

[1] R. R. Coifman and Y. Meyer, *Fourier analysis of multilinear convolution, Calderon theorem and analysis of Lipschitz curves., Euclidean Harmonic Analysis*, proc. Sem. Univ. Maryland, College Park, Md, Lecture Notes in Math. **779**, 104–122, 1979.

[2] Y. Domar, *Harmonic analysis based on certain commutative Banach algebras*, Acta Math. Djursholm **96**, 1–66, 1956.

[3] H. G. Feichtinger and A. T. Gürkanlı, *On a family of weighted convolution algebras*, Int. J. Math. Math. Sci. **13**, 517–526, 1990.

[4] R. H. Fischer, A. T. Gürkanlı and T. S. Liu, *On a family of weighted space*, Math. Slovaca **46**, 71–82, 1996.

[5] G.I. Gaudry, *Multipliers of weighted Lebesgue and measure spaces*, Proc. London Math. Soc. **3 (19)**, 327–340, 1969.

[6] L. Grafakos and R. Torres, *Multilinear Calderon-Zygmund theory*, Adv. Math. **165**, 124–164, 2002.

[7] K. Gröchenig, *Weight functions in time-frequency analysis*, ArXiv: math/0611174v1i, Math.FA, 2006.

[8] Ö. Kulak and A. T. Gürkanlı, *Bilinear multipliers of weighted Lebesgue spaces and variable exponent Lebesgue spaces*, J. Inequal. Appl. **2013**, 2013; Article ID: 259.

[9] G. N. K. Murty and K. R. Unni, *Multipliers on weighted spaces*, In: Functional Analysis and Its Applications, (International Conference), Lecture Notes in Mathematics, **399**, Springer, Berlin, 272–291, 1973.

[10] H. Reiter, *Classical Harmonic analysis and locally compact groups*, Oxford University Press, Oxford, 1968.

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New type of q-Bernstein operators

Pembe Sabancıgil

Bernstein polynomials have a significant role in approximation theory and also in the other fields of mathematics. With the rapid development of q-calculus, Bernstein polynomials based on the q-integers were firstly introduced by Lupas [2] in 1987 and another generalization of Bernstein polynomials based on the q-integers was introduced by Phillips [5] in 1996. The q-Bernstein polynomials quickly gained the popularity and then many operators based on the q-integers were introduced and examined by some other authors. First of all, we give some notations and definitions of q-calculus. For any non-negative integer n, the q-integer of the number n is defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q} & if \quad q \neq 1 \\ n & if \quad q = 1 \end{cases} \quad \text{where } q \text{ is a positive real number.}$$

The q-factorial is defined by

$$[n]_q! = [1]_q [2]_q \ldots [n]_q \text{ and } [0]_q! = 1.$$

For integers $0 \leq k \leq n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[k]_q![n-k]_q!} & if \quad q \neq 1 \\ \begin{pmatrix} n \\ k \end{pmatrix} & if \quad q = 1 \end{cases}.$$

In the present study, we introduce a new type of Bernstein operators based on the q-integers as follows:

Let $0 < q < 1$ and $n \in \mathbb{N}$. For $f : [0,1] \to \mathbb{R}$ and $x \in (0,1)$

$$B^*_{n,q}(f;x) = \frac{[n]_q}{x(1-x)} \sum_{k=0}^{n} p_{n,k}(q,x) f\left(\frac{[k]_q}{[n]_q}\right) \left(\frac{[k]_q}{[n]_q} - x\right)^2,$$

where $p_{n,k}(q,x) = x^k \prod_{j=0}^{n-k-1} (1 - q^j x) \begin{bmatrix} n \\ k \end{bmatrix}_q$.

We calculate the moments of these operators, $B^*_{n,q}(t^j;x)$ for $j = 0,1,2$ and the second order central moment $B^*_{n,q}((t-x)^2;x)$. We estimate the rate of convergence for continuous functions. Furthermore, we prove a local approximation theorem in terms of second modulus of continuity, we obtain a local direct estimate for the new type q-Bernstein operators in terms of Lipschitz type maximal function of order β and we prove a direct global approximation theorem by using the Ditzian-Totik modulus of second order.

2020 MSC: 41A25, 41A36, 47A58

KEYWORDS: q-calculus, q-Bernstein polynomials, Moments

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References

[1] S. N. Bernstein, *Démostration du théoréme de Weierstrass fondée sur le calcul de probabilités*, Communications of the Kharkov Mathematical Society **13 (2)**, 1–2, 1912.

[2] A. Lupaş, *A q-analogue of the Bernstein operator*, University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus **9**, 1987.

[3] N. I. Mahmudov, *The moments for q-Bernstein operators in the case* $0 < q < 1$, Numerical Algorithms **53**, 439–450, 2010; DOI: 10.1007/s11075-009-9312-1.

[4] S. Ostrovska, *The first decade of the q-Bernstein polynomials: Results and perspectives*, J. Math. Anal. Approx. Theory **2**, 35–51, 2007.

[5] G. M. Phillips, *On generalized Bernstein polynomials*, Numerical Analysis 263–269, World Sci.Publ., River Edge, NJ, 1996.

[6] G. M. Phillips, *Bernstein polynomials based on the q-integers*, Annals of Numer. Math. **4**, 511–518, 1997.

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Results on some Sheffer polynomials

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In this study, we investigate the Narumi polynomials and the Pidduck polynomials belonging to the family of the Sheffer polynomials. We give some properties of these polynomials by using by using the methods of the umbral calculus.

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KEYWORDS: Umbral calculus, Sheffer polynomials, Narumi polynomials, Pidduck polynomials, Generating functions

Introduction

Throughout of this paper, we can use the following notations and definitions, which are given by Roman [9, pp. 1-125].

Let P be the algebra of polynomials in the single variable x over the field complex numbers. Let P^* be the vector space of all linear functionals on P. Let $\langle L \mid p(x)\rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let $\mathfrak{F}$ denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k.$$

Such algebra is called an umbral algebra. Each $f \in \mathfrak{F}$defines a linear functional on P and

$$a_k = \left\langle f(t) \mid x^k \right\rangle$$

for all $k \geqslant 0$.

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $o(f(t)) = 1$ will be called a delta series. When we are considering a delta series $f(t)$ in $\mathfrak{F}$ as a linear functional we will refer to it as a delta functional.

It is well-known that $\left\langle t^k \mid x^n \right\rangle = n!\delta_{n,k}$ where δ denotes Kronecker symbol. For all $f(t)$ in $\mathfrak{F}$

$$f(t) = \sum_{k=0}^{\infty} \frac{\left\langle f(t) \mid x^k \right\rangle}{k!} t^k.$$

By a sequence $s_n(x)$ of polynomials we shall imply that $\deg s_n(x) = n$.

Theorem 1 (*cf.* [9, Theorem 2.3.6, p. 20])**.** *Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exist a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions*

$$\left\langle g(t) f(t)^k \mid s_n(x) \right\rangle = n!\delta_{n,k}$$

for all $n, k \geq 0$.

The sequence $s_n(x)$ in (3) is the Sheffer polynomials for pair $(g(t), f(t))$, where $g(t)$ must be invertible and $f(t)$ must be delta series.

The Sheffer polynomials for pair $(g(t), t)$ is the Appell polynomials or Appell sequences for $g(t)$. The Sheffer polynomials for pair $(1, f(t))$ is the associated sequence for $f(t)$ and $s_n(x)$ is associated to $f(t)$.

The generating function of Sheffer polynomials is

$$\frac{1}{g\left(\overline{f}\left(t\right)\right)}e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k\left(y\right)}{k!}t^k, \tag{1}$$

where $\overline{f}\left(t\right)$ is the compositional inverse of $f(t)$.

The Sheffer polynomials satisfy the following relations:

$$s_n\left(x\right) = g(t)^{-1}p_n\left(x\right), \tag{2}$$

$$f\left(t\right)S_n\left(x\right) = nS_{n-1}\left(x\right), \tag{3}$$

$$s_n\left(x+y\right) = \sum_{k=0}^{n}\binom{n}{k}p_{n-k}\left(y\right), \tag{4}$$

where $p_n\left(x\right)$ is associated to $f\left(t\right)$.

Let be $p\left(x\right) \in P$ then, operator e^{yt} satisfy following property:

$$e^{yt}p\left(x\right) = p\left(x+y\right) \tag{5}$$

(*cf.* [9]).

The Narumi polynomials $N_n\left(x\right)$ are Sheffer for the pair

$$g\left(t\right) = \frac{e^t-1}{t},$$

$$f\left(t\right) = e^t - 1.$$

(*cf.* [2, 7, 9] and [8]).

The Pidduck polynomials $P_n\left(x\right)$ are Sheffer for the pair

$$g\left(t\right) = \frac{2}{e^t+1},$$

$$f\left(t\right) = \frac{e^t-1}{e^t+1},$$

(*cf.* [2, 7, 9] and [8]).

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This paper is dedicated to Professor Taekyun KIM on the occasion of his 60th anniversary.

References

[1] P. Blasiak, G. Dattoli, A. Horzela and K. A. Penson, *Representations of monomiality principle with Sheffer-type polynomials and boson normal ordering*, Phys. Lett. A **352**, 7–12, 2006.

[2] R. P. Boas and R. C. Buck, *Polynomial expansions of analytic functions*, Springer Berlin, Heidelberg, 1958.

[3] A. D. Buacchianico and D. Loeb, *A selected survey of umbral calculus*, The Electronic Journal of Combinatorics, 2000.

[4] G. Dattoli, M. Migliorati and H. M. Srivastava, *Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials*, Math. Comput. Modelling **45**, 1033–1041, 2007.

[5] R. Dere and Y. Simsek, *Genocchi polynomials associated with the Umbral algebra*, Appl. Math. Comput. **218 (3)**, 756–761, 2011.

[6] R. Dere and Y. Simsek, *Hermite base Bernoulli type polynomials on the umbral algebra*, Russ. J. Math. Phys. **22 (1)**, 1–5, 2015.

[7] A. Erdelyi, *Higher transcendental functions*, The Bateman Manuscript Procect, Vols I-III. McGraw-Hill, New York, 1953.

[8] T. Kim and D. S. Kim, *Some identities involving associated sequences of special polynomials*, J. Comput. Anal. Appl. **16**, 626–642, 2014.

[9] S. Roman, *The umbral calculus*, Dover Publ. Inc. New York, 2005.

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Capacity solutions for nonlinear parabolic-elliptic systems with degenerate conductivities

Rabab Elarabi

In this work, we investigate the existence of a capacity solution for a coupled parabolic-elliptic system. This system describes the evolution of temperature u and electric potential φ in a semiconductor material. The equations involve nonlinearity g, a divergence constraint on φ, and specific boundary and initial conditions. We apply this study to generalized Orlicz spaces, which may not be reflexive. The nonlinearity g adheres to natural growth and sign conditions.

2020 MSC: 11XX, 35J70

KEYWORDS: Perturbed Parabolic-elliptic system, Musielak-Orlicz-Sobolev spaces, Weak solutions, capacity solutions

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References

[1] S. N. Antontsev and M. Chipot, *The thermistor problem: existence, smoothness, uniqueness, blowup,* SIAM J. Math. Anal. **25**, 1128–1156, 1994.

[2] A. Benkirane and M. Sidi El Vally (Ould Mohamedhen Val), *Some approximation properties in Musielak-Orlicz-Sobolev spaces*, Thai. J. Math. **10 (2)**, 371–381, 2012.

[3] A. Benkirane, J. Douieb and M. Ould Mohamedhen Val. *An approximation theorem in Musielak-Orlicz-Sobolev spaces*, Comment. Math. **51 (1)**, 109–120, 2011.

[4] R. Elarabi, *Strongly nonlinear coupled system in Orlicz–Sobolev spaces without 12-condition*, J. Elliptic Parabol. Equ. **8 (1)**, 77–106, 2022.

[5] J. P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Analysis **11**, 379–392, 1987.

[6] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly or slowly increasing coefficients,* Trans. Amer. Math. Soc. **190**, 163–205, 1974.

[7] M. T. González Montesinos and F. Ortegón Gallego, *The evolution thermistor problem with degenerate thermal conductivity.* Commun. Pure Appl. Anal., **1 (3)**, 313–325, 2002.

[8] M. T. González Montesinos and F. Ortegón Gallego, *Existence of a capacity solution to a coupled nonlinear parabolic-elliptic system.* Commun. Pure Appl. Anal., **6 (1)**, 23–42, 2007.

[9] J. Musielak, *Orlicz Spaces and Modular Spaces,* Lecture Notes in Mathematics (Volume 1034), Springer, Berlin, 1983.

[10] K. R. Rajagopal and M. Rŭžička, *Mathematical modeling of electrorheological materials*, Contin. Mech. Thermodyn. **13**, 59–78, 2001.

[11] M. Rŭžička, *Electrorheological fluids: Modeling and mathematical theory*, Lecture Notes in Mathematics, Springer, Berlin, 2000.

[12] X. Xu, *A strongly degenerate system involving an equation of parabolic type and an equation of elliptic type*, Comm. Partial Differential Equations **18**, 199–213, 1993.

[13] X. Xu, *A degenerate Stefan-like problem with Joule's heating.* SIAM J. Math. Anal. **23**, 1417–1438, 1992.

[14] X. Xu, *On the existence of bounded temperature in the thermistor problem with degeneracy*, Nonlinear Anal., Theory Methods Appl. **42**, 199–213, 2000.

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Eta Dedekind functions associated to Legendre symbols

Abdelmejid Bayad [1] *and Sofiane Atmani* [*2]

Let

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = \exp(2\pi i \tau), \quad \mathrm{Im}(\tau) > 0$$

be the Dedekind eta-function. Let p be a rational prime number. Then the Fourier coe cients of the eta product $\eta(p\tau)^p / \eta(\tau)$ are non-negative.

For p an odd prime define

$$\chi_p(m) = \left(\frac{m}{p}\right) \quad \text{(the Legendre symbol).}$$

Suppose k is an integer, $k \geq 2$, and $(p-1)/2 \equiv k\,(mod 2)$. Define the Eisenstein series

$$E_{p,k}(q) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi_p(m)\, n^{k-1} q^{mn}.$$

Then $E_{p,k}$ is a modular form of weight k and character χ_p for the congruence subgroup $\Gamma_0(p)$. See [8, 10] for more details.

For any prime p, in this talk we are interested by the connection between the Eisenstein series $E_{p,k}(q)$ and the modulra forms $\eta(p\tau)^p / \eta(\tau)$.

Among others, our motivation comes from the following identity found by Ramanujan:

$$U_{5,2} = \frac{\eta(5\tau)^5}{\eta(\tau)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi_5(m)\, n q^{mn}.$$

Kolberg [10] has found many relations between such Eisenstein series and certain eta products. The eta function $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$. Hence the modular forms

$$\frac{\eta(5\tau)^5}{\eta(\tau)}, \qquad \frac{\eta(\tau)^5}{\eta(5\tau)}$$

are modular forms of weight 2, on $\Gamma_0(p)$ with character $\left(\frac{\cdot}{5}\right)$. We have the important relation

$$\begin{aligned} E_{5,6}(q) &= \eta(5\tau)^3 \eta(\tau)^9 + 40\eta(5\tau)^9 \eta(\tau)^3 + 335 \frac{\eta(5\tau)^{15}}{\eta(\tau)^3} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi_5(m)\, n^5 q^{mn}. \end{aligned}$$

Incidentaly, this study has many consequence on the problem of the Ramanujan congruences modulo power of primes for various partitions functions.

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KEYWORDS: Eisenstein series, Eta Dedekind function, Legendre symbols

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References

[1] S. Atmani, A. Bayad and M-O. Hernane. *Congruences for Fourier coefficients of eta-quotients modulo powers of 5; 7; 11; 13, and 17*, Math. Methods Appl. Sci. **46 (5)**, 5001–5028, 2022.

[2] A. Bayad and A. Raouj, *Mean values of L-functions and Dedekind sums'*, J. Number Theory **132**, 1645–1652, 2012.

[3] N. J. Fine, *On a system of modular functions connected with the Ramanujan identities*, Tohoku Math. J. **8 (2)**, 149–164, 1956.

[4] F. G. Garvan, *A simple proof of Watson's partition congruences for powers of 7*, J. Austral. Math. Soc. Ser. A **36 (3),** 316–334, 1984.

[5] F. G. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod 5; 7 and 11*, Trans. Amer. Math. Soc. **305 (1)**, 47–77, 1988.

[6] F. Garvan, *Cubic modular identities of Ramanujan, hypergeometric functions and analogues of the arithmetic-geometric mean iteration*, In: The Rademacher legacy to mathematics (University Park, PA, 1992), volume 166 of Contemp. Math., pages 245{264. Amer. Math. Soc., Providence, RI, 1994.

[7] F. Garvan, D. Kim and D. Stanton, *Cranks and t-cores*, Invent. Math. **101 (1),** 1–17, 1990.

[8] F. Garvan, *Some congruences for partitions that are p-cores*, Proc. London Math. Soc. **66(3),** 449–478, 1993.

[9] A. Granville and K. Ono, *Defect zero p-blocks for nite simple groups,* Trans. Amer. Math. Soc. **348 (1)**, 331–347, 1996.

[10] O. Kolberg, *Note on the Eisenstein series of* $\gamma_0\,(p)$, Arbok Univ. Bergen Mat.-Natur. Ser. **6**, 1969, 1968.

[11] F. G. Garvan, *A simple proof of Watsons partition congruences for powers of 7*, J. Austral. Math. Soc. Ser. A **36 (3),** 316–334, 1984.

[12] H. Hida, *Elementary theory of L-functions and Eisenstein series*, Volume 26 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1993.

[13] F. G. Garvan, *Congruences for Andrews spt-function modulo powers of 5, 7 and 13*, Trans. Amer. Math. Soc. **364 (9)**, 4847–4873, 2012.

[14] F. G. Garvan, D. Kim and D. Stanton, *Cranks and t-cores*, Invent. Math. **101 (1)**, 1–17, 1990.

[15] B. Gordon, *Ramanujan congruences for* p_k $(mod\ 11^r)$, Glasgow Math. J. **24 (2)**, 107–123, 1983.

[16] N. J. Fine, *On a system of modular functions connected with the Ramanujan identities*, Thoku Math. J. **8 (2)**, 149–164, 1956.

[17] S. Ramanujan, *Congruence properties of partitions*, Math. Z. **9**, 147–153, 1921.

[18] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n*, Proc. Cambridge Philos. Soc. **19**, 207–210, 1919; Collected papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 210–213, 2000.

[19] L. Wang, *Congruences for 5-regular partitions modulo powers of 5*, Ramanujan J. **44 (2)**, 343–358, 2017.

[20] L. Wang, *Congruences modulo powers of 11 for some partition functions*, Proc. Am. Math. Soc. **146 (4)**, 1515–1528, 2018.

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Probabilistic Lah numbers and Lah-Bell polynomials

Siqi Dong *[1], *Yuankui Ma* [2], *Taekyun Kim* [3] *and Dae San Kim* [4]

Let Y be a random variable whose moment generating function exists in some neighborhood of the origin. We study the probabilistic Lah numbers associated with Y and the probabilistic Lah-Bell polynomials associated with Y, as probabilistic versions of the Lah numbers and the Lah-Bell polynomials, respectively. We derive some properties, explicit expressions, recurrence relations and certain identities for those numbers and polynomials. In addition, we treat the special cases that Y is the Poisson random variable with parameter $\alpha > 0$ and the Bernoulli random variable with probability of success p.

2020 MSC: 11B73, 11B83

KEYWORDS: Probabilistic Lah numbers, probabilistic Lah-Bell polynomials

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QSPR investigation of certain molecular descriptor and monocarbolic acids

Sushmitha Jain [*1] *and Veerabhadraiah Lokesha* [2]

The analysis of QSPR contributes a substantial structural intuition into the physio-chemical properties of monocarbolic acids. This investigates some physio-chemical properties of monocarbolic acids and elaborate a QSPR model using few topological indices and monocarbolic acids. Here we analyze how extent the topological indices are connected to the physio-chemical properties of monocarbolic acids. Therefore, we compute analytically the topological indices of monocarbolic acids and plot the graphs between each of these topological indices to the properties of monocarbolic acids. This QSPR model reveal a close correlation between heavy atomic count, complexity index of refraction and molecular weight of monocarbolic acids with some of the most successful topological indices.

2020 MSC: 5C05, 05C12

KEYWORDS: QSPR, Topological index, Monocarbolic acids

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Generalized type 2 degenerate the Euler-Genocchi polynomials

Si-Hyeon Lee

Recently, Kim-Kim-Kim introduced generalized degenerate Euler-Genocchi polynomials. From this idea we consider generalized type 2 Euler-Genocchi polynomials as a degenerate version. In this paper, we study some properties and identities of the generalized type 2 degenerate Euler-Genocchi polynomials. In addition, it was expressed as an equation using the Fermionic integral.

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KEYWORDS: Degenerate Euler-Genocchi polynomials, Fermionic integral

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Exploring λ-Sheffer sequences: Representations and transformations within λ-umbral calculus

Seongho Park

This paper presents a study on the representation of λ-Sheffer polynomials through other λ-Sheffer polynomials. Building upon recent advancements in λ-umbral calculus, we explore the process of substituting the traditional exponential function with a degenerate exponential function in the generating functions of Sheffer polynomials. Through this approach, we derive new methods to represent λ-Sheffer polynomials using other λ-Sheffer polynomials. Utilizing formulas derived from the definition of λ-Sheffer polynomials, this work elucidates the relationships among these polynomials and provides a deeper analysis of the properties of degenerate polynomials and sequences. This research deepens the understanding of λ-umbral calculus and offers new perspectives on the theory and applications of degenerate polynomials.

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KEYWORDS: Sheffer sequences, Umbral calculus, Bernoulli numbers, Degenerate polynomials

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References

[1] S. Araci, *Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus*, Appl. Math. Comput. **233**, 599–607, 2014.

[2] J. Brillhart, *Mathematical notes: Note on the single variable Bell polynomials*, Amer. Math. Monthly **74 (6)**, 695–696, 1967.

[3] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15**, 51–88, 1979.

[4] L. Carlitz, *A degenerate Staudt-Clausen theorem*, Arch. Math. (Basel) **7**, 28–33, 1956.

[5] R. Dere and Y. Simsek, *Applications of umbral algebra to some special polynomials*, Adv. Stud. Contemp. Math. (Kyungshang) **22 (3)**, 433–438, 2012.

[6] R. Dere and Y. Simsek, *Hermite base Bernoulli type polynomials on the umbral algebra*, Russ. J. Math. Phys. **22 (1)**, 1–5, 2015.

[7] L.-C. Jang, D. S. Kim, H. Kim, T. Kim and H. Lee, *Study of degenerate poly-Bernoulli polynomials by λ-umbral calculus*, CMES-Computer Modeling in Engineering and Sciences **129 (1)**, 393–408, 2021; DOI:10.32604/cmes.2021.016917.

[8] H. K. Kim, *Degenerate Lah-Bell polynomials arising from degenerate Sheffer sequences*, Adv. Difference Equ. **2020**, 2020; Article ID: 687.

[9] D. S. Kim and T. Kim, *A Note on a new type of degenerate Bernoulli numbers*, Russ. J. Math. Phys. **27 (2)**, 227–235, 2020.

[10] D. S. Kim and T. Kim, *Degenerate Sheffer sequences and λ-Sheffer sequences*, J. Math. Anal. Appl. **493 (1)**, 2021; Article ID: 124521.

[11] T. Kim and D. S. Kim, *Some identities on truncated polynomials associated with degenerate Bell polynomials*, Russ. J. Math. Phys **28 (3)**, 342–355, 2021.

[12] T. Kim and D. S. Kim, *Note on the degenerate gamma function*, Russ. J. Math. Phys. **27 (3)**, 352–358, 2020.

[13] T. Kim, D. S. Kim, D. V. Dolgy and J.-W. Park, *On the type 2 poly-Bernoulli polynomials associated with umbral calculus*, Open Math. **19 (1)**, 878–887, 2021.

[14] T. Kim, D. S. Kim, H.-Y. Kim, H. Lee and L.-C. Jang, *Degenerate poly-Bernoulli polynomials arising from degenerate polylogarithm*, Adv. Difference Equ. **2020**, 2020; Article ID: 444.

[15] T. Kim, D. S. Kim, J. Kwon and H. Lee, *Representations of degenerate poly-Bernoulli polynomials*, J. Inequal. Appl. **2021**, 2021; Article ID: 58.

[16] T. Kim, D. S. Kim, H. Lee and J. Kwon, *A note on some identities of new type degenerate Bell polynomials*, Mathematics **7 (10)**, 2019; Article ID: 1086.

[17] Y. Ma and T. Kim, *A note on negative λ-binomial distribution*, Adv. Difference Equ. **2020**, 2020; Article ID: 569.

[18] Y. Ma, D. S. Kim, H. Lee and T. Kim, *Poly-Dedekind sums associated with poly-Bernoulli functions*, J. Inequal. Appl. **2020**, 2020; Article ID: 248.

[19] S. Roman, *The umbral calculus*, Pure and Applied Mathematics 111, Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers), New York, 1984.

[20] S. Roman, *More on the umbral calculus with emphasis on the q-umbral calculus*, J. Math. Anal. Appl. **107 (1)**, 222–254, 1985.

[21] Y. Simsek, *Identities on the Changhee numbers and Apostol-type Daehee polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **27 (2)**, 199–212, 2017.

[22] Y. Simsek, *Special numbers and polynomials including their generating functions in umbral analysis methods*, Axioms **7 (2)**, 2018; Article ID: 22.

[23] E. T. Whittaker and G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions*, Reprint of the fourth edition 1927; Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996.

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Some explicit formulae involving Hardy sums

Mu Yaya [1] *and Zhang Tianping* [*1,2]

Various properties of the Hardy sums were investigated by many authors. One aim of this paper is to present a method that can remedy the previous flaws (for example, [Some identities involving certain Hardy sum and Kloosterman sum, Journal of Number Theory, 2016; New identities involving Hardy sums $S_3(h,k)$ and general Kloosterman sums, AIMS Mathematics, 2021; Some novel identities for analogues of Dedekind sums, Hurwitz zeta-function and general Kloosterman sum, Acta Mathematica Hungarica, 2022]), in studying the hybrid sums involving various Hardy sums and general Kloosterman sums. Another one is to answer a question previously posed in [New identities involving Hardy sums $S_3(h,k)$ and general Kloosterman sums, AIMS Mathematics, 2021]. By overcoming some technical obstacles, several explicit formulae are given. This is a joint work with Yaya Mu.

2020 MSC: 11F20, 11L05

Keywords: Hardy sums, general Kloosterman sums, Gauss sums, explicit formula.

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The new type degenerate Fubini polynomials

Uitae Pyo

Recently, T. Kim has investigated degenerate Fubini polynomials, revealing various theorems and diverse relationships with Euler polynomials. In this paper, we extend the exploration of properties associated with additional degenerate Fubini polynomials. We introduce new types of degenerate Fubini polynomials and examine their properties. Additionally, we will explore properties related to Bernoulli polynomials and estimate polynomial values by substituting specific values.

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KEYWORDS: Euler polynomials, Fubini polynomials, Degenerate Fubini polynomials

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The moment generating function related to Bell polynomials

Uitae Pyo

In this paper, an investigation has been conducted on the moment functions of various probability variables concerning the recently studied exponential λ-analogue. Using degenerate polynomials, generalized expressions for the mean and variance were derived and directly computed. Additionally, by leveraging Stirling numbers and Bell polynomials, connections with Poisson random variables were explored. Furthermore, meaningful results were obtained using covariance analysis. Moreover, we utilized these results to define a new form of Bell polynomials. We anticipate that investigating the properties of these polynomials in future research will yield new and valuable results.

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KEYWORDS: Bell polynomials, Stirling numbers, Generating function

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A study on the generalized degenerate multi-Bernoulli polynomials

Wonjoo Kim *[1] *and Jongkyum Kwon* [2]

Recenerately, Kim-Kim-Kim (2022) studied the degenerate multi-Euler-Genocchi polynomials as degenerate versions of some special polynomials. In this paper, we consider the multi-Bernoulli polynomials, generalized degenerate multi-Bernoulli polynomials and investigate some identities and properties of them.

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KEYWORDS: Bernoulli polynomials, Euler-Genocchi polynomials

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On the Dedekind sums and its a new reciprocity formula

Wenpeng Zhang

To describe the results of this paper, we first need to introduce the definition of the Dedekind sums $S(r,q)$. For any integers $q \geq 2$ and r, the classical Dedekind sums $S(r,q)$ is defined as follows (see [2]):

$$S(r,q) = \sum_{c=1}^{q} \left(\left(\frac{c}{q}\right)\right)\left(\left(\frac{rc}{q}\right)\right),$$

where as usual, $((u))$ is defined as

$$((u)) = \begin{cases} u - [u] - \frac{1}{2}, & \text{if } u \text{ is not an integer;} \\ 0, & \text{if } u \text{ is an integer.} \end{cases}$$

Usually, we know that $S(r,q)$ describes the behaviour of the logarithm of the eta-function (see [15] and [16]) under modular transformations. Since the importance of $S(r,q)$ in analytic number theory, many authors have studied the various arithmetical properties of $S(r,q)$, and obtained a series of interesting results, some of them can be found in the references [3]- [12]. It is worth mentioning that perhaps the most important property of $S(r,q)$ is its reciprocity theorem (see [2, 3, 13]). That is, for any positive integers u and v with $(u,v)=1$, one has the identity

$$S(u,v) + S(v,u) = \frac{u^2+v^2+1}{12uv} - \frac{1}{4}. \tag{1}$$

H. Rademacher and E. Grosswald [16] also obtained a three-term formula similar to (1).

The main purpose of this paper is using the analytic methods and the properties of Dirichlet L-functions to study the properties of $S(r,q)$, and give a new reciprocity formula for it. That is, we will prove the following three conclusions.

Theorem 1. *Let h and q be two positive odd numbers with $(h,q)=1$. Then we have the reciprocity formula*

$$S\left(\overline{2}q, h\right) + S\left(\overline{2}h, q\right) = \frac{h^2+q^2+4}{24hq} - \frac{1}{4},$$

where $\overline{2}$ in $S\left(\overline{2}h, q\right)$ and $S\left(\overline{2}q, h\right)$ are $\frac{q+1}{2}$ and $\frac{h+1}{2}$, respectively.

As an application of Theorem 1, we can also deduce a new calculating formula for the mean square value of Dirichlet L-functions with the weight of the character sums. That is, we have

Theorem 2. *For any positive integer $q>1$ and $(q,6)=1$, we have the identities*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(3)\cdot\overline{\chi}(2)\cdot|L(1,\chi)|^2$$
$$= \frac{\pi^2}{18}\cdot\frac{\phi^2(q)}{q^2}\cdot\left[\frac{q}{4}\cdot\prod_{p|q}\left(1+\frac{1}{p}\right) - \frac{9}{2} + \left(\frac{q}{3}\right)\cdot\prod_{p|q}\frac{p-\left(\frac{p}{3}\right)}{p-1}\right],$$

where $\sum\limits_{\substack{\chi \bmod q \\ \chi(-1)=-1}}$ *denotes the summation over all odd characters modulo* q*,* $L(s,\chi)$ *denotes the Dirichlet* L*-function corresponding to character* $\chi \bmod q$*,* $\phi(q)$ *is the Euler function,* $\prod\limits_{p|q}$ *denotes the product over all distinct prime divisors of* q*, and* $\left(\frac{*}{3}\right)$ *is the Legendre's symbol modulo* 3*.*

Theorem 3. *For any positive integer* $q>1$ *with* $(q,6)=1$*, we have the identity*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}} |L(1,\chi\lambda_3)|^2 = \frac{\pi^2}{27}\cdot\frac{\phi^2(q)}{q^2}\left[2q\cdot\prod_{p|q}\left(1+\frac{1}{p}\right)-\left(\frac{q}{3}\right)\cdot\prod_{p|q}\frac{p-\left(\frac{p}{3}\right)}{p-1}\right],$$

where $\sum\limits_{\substack{\chi \bmod q \\ \chi(-1)=1}}$ *denotes the summation over all even characters modulo* q*, and* $\lambda_3 = \left(\frac{*}{3}\right)$ *denotes the Legendre's symbol modulo* 3*.*

Some notes: It is clear that replacing 3 with 5 or 7 in Theorem 2 we can also get some similar results, but the situation is more complicated and we do not list them.

From the reciprocity formula (1) and the Lemma 2 in [20] we may immediately deduce that for any two distinct odd primes p and q, one has the identity

$$\frac{q}{q-1}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}\chi(p)\cdot|L(1,\chi)|^2+\frac{p}{p-1}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}\chi(q)\cdot|L(1,\chi)|^2 = \frac{\pi^2}{12}\cdot\frac{p^2+q^2-3pq+1}{pq}. \tag{2}$$

Whether there exists a direct proof of (2) (only use the properties of Dirichlet L-function, the reciprocity formula (1) can not be used) is an open problem.

In addition, Theorem 3 is an interesting result. In fact, for the mean square value of Dirichlet L-functions with even characters at point $s=1$, there are so far only various asymptotic formulas, without any exact identities. Theorem 3 gives an exact calculating formula for it, just by turning all characters χ with $\chi(-1)=1$ into $\chi\lambda$ with $\chi\lambda(-1)=-1$.

2020 MSC: 11F20, 11F68

KEYWORDS: Dedekind sums, Dirichlet L-function

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Dedicated to Professor Taekyun Kim on the occasion of his sixtieth birthday.

References

[1] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976.

[2] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Springer-Verlag, New York, 1976.

[3] L. Carlitz, *The reciprocity theorem for Dedekind sums*, Pacific J. Math. **3**, 523–527, 1953.

[4] Y. W. Chen, N. Dunn and S. Silas, *Dedekind sums $S(a,b)$ and inversions modulo b*, Int. J. Number Theory **11**, 2325–2339, 2015.

[5] J. B. Conrey, E. Fransen, R. Klein and C. Scott, *Mean values of Dedekind sums*, J. Number Theory **56**, 214–226, 1996.

[6] K. Girstmair, *On Dedekind sums with equal values*, Int. J. Number Theory **12**, 473–481, 2016.

[7] K. Girstmair, *On a recent reciprocity formula for Dedekind sums*, Int. J. Number Theory **15**, 1469–1472, 2019.

[8] K. Girstmair, *On the fractional parts of Dedekind sums*, Int. J. Number Theory **11**, 29–38, 2015.

[9] D. Han and W. P. Zhang, *Some new identities involving Dedekind sums and the Ramanujan sum*, Ramanujan J. **35**, 253–262, 2014.

[10] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, Springer-Verlag, New York, 1982.

[11] W. Kohnen, *A short note on Dedekind sums*, Ramanujan J. **45**, 491–495, 2018.

[12] S. Macourt, *An integer-valued expression of Dedekind sums*, Int. J. Number Theory **13**, 1253–1259, 2017.

[13] L. J. Mordell, *The reciprocity formula for Dedekind sums*, Amer. J. Math. **73**, 593–598, 1951.

[14] Y. N. Liu and W. P. Zhang, *A hybrid mean value related to the Dedekind sums and Kloosterman sums*, Acta Math. Sin., Engl. Ser. **27**, 435–440, 2011.

[15] H. Rademacher, *On the transformation of* $\log \eta(\tau)$, J. Indian Math. Soc. **19**, 25–30, 1955.

[16] H. Rademacher and E. Grosswald, *Dedekind sums*, Carus Mathematical Monographs, Math. Assoc. Amer., Washington D. C., 1972.

[17] E. Tsukerman, *Fourier-Dedekind sums and an extension of Rademacher reciprocity*, Ramanujan J. **37**, 421–460, 2015.

[18] W. P. Zhang, *On the fourth power mean of Dirichlet L-functions*, Lecture Notes in Contemporary Mathematics, Science Press, Beijing, 173–179, 1989.

[19] W. P. Zhang, *On a note of one class mean square value of L-functions*, Journal of Northwest University **20**, 9–12, 1990.

[20] W. P. Zhang, *On the mean values of Dedekind Sums*, Journal de Théorie des Nombres de Bordeaux **8**, 429–442, 1996.

[21] W. P. Zhang, *A sum analogous to Dedekind sums and its hybrid mean value formula*, Acta Arithemetica **107**, 1–8, 2003.

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Probabilistic degenerate Fubini polynomials associated with random variables

Yuankui Ma [*1], *Rongrong Xu* [2], *Taekyun Kim* [3] *and Dae San Kim* [4]

Let Y be a random variable such that the moment generating function of Y exists in a neighborhood of the origin. The aim of this paper is to study probabilistic versions of the degenerate Fubini polynomials and the degenerate Fubini polynomials of order r, namely the probabilisitc degenerate Fubini polynomials associated with Y and the probabilistic degenerate Fubini polynomials of order r associated with Y. We derive some properties, explicit expressions, certain identities and recurrence relations for those polynomials. As special cases of Y, we treat the gamma random variable with parameters $\alpha, \beta > 0$, the Poisson random variable with parameter $\alpha > 0$, and the Bernoulli random variable with probability of success p.

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KEYWORDS: Probabilistic degenerate Fubini polynomials, probabilistic degenerate Fubini polynomials of order r

Acknowledgments

This paper is dedicated to Professor Taekyun KIM on the occasion of his 60th anniversary.

This work is a joint work which is also on arxiv: 2401.02638.

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To derive a moment inequality involving stochastic integrals using a Gronwall-type inequality

Young-Ho Kim

The main purpose of this presentation is to discuss the Itô formula for the stochastic process and the application of the Gronwall-type inequality to induce the moment inequality of the Itô integral. More specifically, we want to establish some stochastic moment inequalities in the stochastic process by applying the Itô formula and the Gronwall-type inequalities as well as introduce a new proofs of some parts of the Burkholder-Davis-Gundy inequality and induce inverse inequality. Next, we will discuss the importance of moment inequality by introducing a theory that applies these results to the existence theorem of solutions of stochastic differential equations.

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KEYWORDS: Itô integral, Stochastic differential equations

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